

# A Theory of Rule Development

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## Abstract

This paper develops a model with endogenously coarse rules. A principal hires an agent to take an action. The principal knows the optimal state-contingent action, but cannot communicate it perfectly due to communication constraints. The principal can use previously realized states as examples to define rules of varying breadth. We analyze how rules are chosen under several assumptions about how rules can be amended. We explore the inefficiencies that arise and how they depend on the ability to refine rules, the principal's time horizon and patience, and other factors. Our model exhibits path dependence in that the efficacy of rule development depends on the sequence of realizations of the state. We interpret this as providing a foundation for persistent performance differences between similar organizations and explore the role of different delegation structures in ameliorating the effects of bounded communication.

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# 1 Introduction

Firms and bureaucracies cannot function without rules. These rules lead them to take inefficient decisions in some circumstances. One long-recognized reason why organizations will be inefficient is that the world is highly complex and it is practically impossible to provide agents with the information-contingent plans they would need to act optimally. In this paper, we develop a model of endogenously coarse rules due to imperfect communication. The agency problem in our model is not one of incentives and private benefits: we assume the agent follows any rules that are set out for him. The difficulty is instead that it will typically be impossible to describe the first-best decision rule to the agent. We discuss the inefficiencies that arise and how they are related to the assumptions about the environment.

We model rule development as a constrained-optimal process in a world with two types of communication constraints. The first is that the organization's agents must react to events they observe before communicating with the principal. Hence, the agents require contingent plans to execute. The second is that it is impossible for the principal to communicate a complete contingent plan to the agent. A novel element of our model is our analogy-based approach to partial describability: we assume that the only classes of states on which rules can be based are sets of states that are similar to some previously realized state, and we assume that there is a commonly understood notion of distance that allows the breadth of a rule to be a choice variable.

We feel that analogies are important in real-world communication and that our model may be appropriate for thinking about rules in a number of contexts. One reason why analogies may be useful in the real world is that states of nature are very complex objects that would be difficult to describe before the the principal and agent have shared experiences. For example, consider a venture capital firms deciding whether to make a particular investment. In such situations assessments are often made relative to previous firms in which they have invested, e.g. “we’d love to find another company like COMPANY X” or “our investment in COMPANY Y was a disaster – we’re avoiding any company that is at all similar.” Prior experiences can also lead them to develop coarse rules like: “we don’t

invest in INDUSTRY X”, or “we only invest if the entrepreneur has had previous success”. Hiring or admissions decisions could be another example. Again, it is common to hear candidates compared to previous candidates who have worked out well or poorly, and rules develop like “don’t hire anyone with a reference from PERSON X” or “never admit anyone from a school like SCHOOL X if they don’t have a good math GRE score.” Such rules are made with reference to a shared experience, they are coarse, they develop over time, and they seem to be attempts to cope with the impossibility of describing a very complex state space.

Kellogg (2008) describes another potential application and provides empirical evidence. He examines the experience of oil production companies and drilling contractors and finds evidence that producer-driller pairs become more efficient over the course of their relationship. A source of these economies may be that shared experience lets the producer describe measures that should be undertaken in particular circumstances to avoid problems that have occurred in the past. Our model might also be applied in a legal context. There it would describe a situation in which precedents are set in particular cases and the accompanying decisions provide some measure of the breadth of situations around the precedent-setting case in which the same reasoning is supposed to apply.

Section 2 describes our model. There are two players: a principal and an agent. There is a discrete set of time periods. In each period a new state of the world arises. The agent observes the state and must take an action. Inefficient decisions are sometimes taken not because of any incentive issues, but because of one of our communication constraints: the agent must act before he can communicate with the principal (and has no prior knowledge of the mapping from states to optimal actions). We assume that the agent will always follow any rule that has been communicated to him, but these will typically be incomplete and/or suboptimal because of our other communication constraint: the principal is restricted to the use of analogy-based rules specifying that an action be taken whenever the new state is within some distance of a previously realized state. The principal does observe the state and has an opportunity to promulgate new rules at the end of each period, so rulebooks will become more complex and efficient over time. We often compare outcomes under three different assumptions about the extent to which rules can be changed over time. We call

these the no overwriting, incremental overwriting, and vast overwriting cases.

Section 3 discusses the simplest case in which rules exist: a two period model in which a rule for second-period behavior can make reference to the first-period state. A primary result is that rules have excess breadth and sometimes produce incorrect decisions. Indeed, at the margin the first-period rule is completely worthless and must produce as many incorrect decisions as correct decisions. Section 4 examines a three period model. A primary observation here is that option-value considerations lead the principal to narrow rules. In the no overwriting case, this may even include declining to expand the breadth of a rule even though it would also be correct in all marginal cases. Section 5 examines the infinite-horizon case. Here, nearly optimal rules must develop eventually if the principal is very patient. Whether exactly optimal rules develop depend on the assumptions about how rules can be revised. We also discuss the distribution and convergence of long-run payoffs for general discount factors.

Section 6 builds on the model to discuss governance structures in firms and how they may evolve. There are now two tasks, the principal has access to two agents, and can also perform a task himself. Time constraints force the principal to choose between delegating both tasks with communication, or performing one task personally and delegating the other without communication. Section 7 develops a connection to the literature on endogenous categorization. Section 8 concludes.

Our paper relates to a number of literatures. Most closely related is the literature on languages and communication within organizations. Arrow (1974, 1984) introduced the notion of a *code* in an organization as “all the known ways, whether or not inscribed in formal rules, for conveying information.” He focuses on the role of codes on returns to scale, and on the irreversible nature of the investment in developing a code. Wernerfelt (2004) analyzes a team-theoretic problem in which agents may use different codes in communicating with a principal in order to optimally deploy an indivisible asset. Cremer et al. (2007) analyze a model with boundedly rational agents where a common language reduces the cost of irrationality, but comes at the expense of being narrowly tailored to the state of the world. They explore the implications of this tradeoff for optimal scope of organizations.

More broadly, there is a long history of studying adaptation in organizations dating

back to Knight (1921), Barnard (1938), and Hayek (1945). Williamson (2005) argues that “adaptation to disturbances [is] the central problem of economic organization”. The largest part of the formal literature examines the principal-agent relationship from incentive theory perspective. Our paper, in contrast, is more similar to the “team theory” approach pioneered by Marschak and Radner (1972) and including such subsequent contributions as Cremer (1980), Bolton and Dewatripont (1994), Wernerfelt (1997), Van Zandt (1999), and Vayanos (2003). We follow this literature in focusing on models in which there are no divergence in preferences between principals and agents. Instead, the fundamental obstacles to efficient organizations are the costs of information processing and communication. Our focus is somewhat different from the much of this literature: whereas many papers in this literature assume a reduced form “cost” of communication and focus on how organizations should be structured to minimize these costs, we spend most of our time characterizing the “costs” that result from the limitations we place on what can and cannot be communicated; and we only get to organizational implications in one section near the end.

Our paper also relates to other literatures that endogenize communication problems in organizations. A large literature following Crawford and Sobel (1982) examines limitations on communication using cheap-talk models. Dessein (2002) notes that such limitations provide a motivation to delegate decisions to agents. Two more recent papers focus more directly on endogenizing communication costs. Dessein and Santos (2006) analyze the tradeoffs between adaptation to changing environments and coordination across agents when communication is imperfect. They emphasize that communication is difficult and include an extension analyzing the quality of communication in a model in which communication can be made more accurate at a cost. Dewatripont and Tirole (2005) focus more directly on endogenizing the limited quality of communication and develop several models featuring differences in preferences, effort costs in the sending and receiving of messages, and various assumptions about the structure of actions. What we are doing is somewhat different from this literature in that we do not have preference differences and focus on the issue of communicating the desired contingent plan-of-action, whereas much of this literature abstracts from this part of the communication problem and simply assumes, for example, that the optimal plan is to set the action “equal” to the signal the agent receives.

Our model is also related to the literature on thinking via categories (Mullainathan (2002), Fryer and Jackson (2007), Mullainathan et al. (2008).) This literature—for the most part—discusses implications of a decision maker having a smaller number of mental categories than is required to perfectly partition events. One way to think of our paper is providing—in a specific context—a foundation for where these categories come from. In our model, the categories are the rules which the principal gives the agent, and they arise as an endogenous response to the bounded communication.

Our discussion of governance structures relates to the much larger theory on why some transactions take place in the market, while others take place inside firms.<sup>1</sup> A complete formal treatment of the tradeoff between the market and the firm in adapting to changed circumstances would involve the following. Fix a particular *transaction* and then compare allocative efficiency in the market and the firm. To do this, however, one needs a model of price formation. Without such a model it seems impossible to talk about degrees of efficiency of the price mechanism. We return to this issue in the final section, but we do not intend to provide such a model in this paper. Instead our section 6 just tackles a more modest task: highlighting a particular friction that occurs within the authority relationship and analyzing how different delegation structures may be used in response to the evolution of this friction. In this regard, our paper is related to Simon (1951), which analyzes a model in which a principal and an agent have preferences over a state contingent action. The state is common knowledge, but contracts are incomplete so that the Pareto efficient state contingent contract is not enforceable. Simon provides conditions under which it is optimal to specify a non-contingent action ex ante rather than allow the principal to mandate an action after observing the state. Baker et al. (2006) introduce cash flow rights which can be allocated ex ante and multiple decisions rights into Simon’s framework in order to analyze how allocation of control can facilitate relational contracting.

Finally, there is an experimental literature on communication in organizations. Most closely related to our paper are Camerer and Weber (2003) and Selten and Warglien (2006). Camerer and Weber (2003) run an experiment involving 16 pictures depicting offices in

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<sup>1</sup>Coase (1937), Klein et al. (1978), Williamson (1971), Williamson (1979), Grossman and Hart (1986), Hart and Moore (1990).

which subjects can use natural language to try to tell each other which picture is relevant in this round. They are interested in the development of codes over time and in how two “firms” with different codes perform when merged. Selten and Warglien (2006) use abstract shapes instead of office pictures, and the available language consists of a few different letters, each with its own cost. The performance-improvements on these tasks are an example of organizational “learning by doing”. MacLeod (2002) develops a boundedly rational model of learning by doing for such settings and fits it to the experimental data. His agents automatically learn the optimal action for each state after it has arisen and also have the opportunity to improve their rules by paying an per-state cost to learn what action will be optimal in some states that have not yet arisen.

## 2 The Model

### 2.1 Statement of the Problem

There are two players: the Principal and the Agent. The Principal hires the Agent to react to idiosyncratic situations and take actions on her behalf in each period  $t = 1, \dots, T$ . At the start of each period, the principal issues a set of rules  $R_t$ . The Agent then observes the state of nature  $\omega \in \Omega$  and chooses an action  $a \in \{-1, 1\}$ . The principal then receives a payoff  $\pi(a, \omega)$  that depends on A’s action and the state of nature. The principal’s payoff in the full game is the discounted sum of his per period payoffs:  $V = \sum_{t=1}^T \delta^t \pi(a_t, \omega_t)$ . We will sometimes write  $V_t \equiv \pi(a_t, \omega_t)$  for the principal’s payoff in period  $t$ .

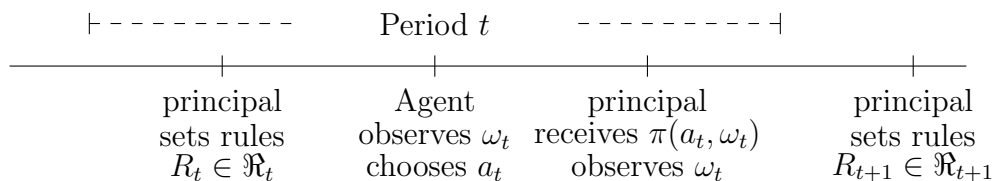


Figure 1: Model timing

There are no hidden action/private benefit issues in our model. The first best would obtain if the principal at each time  $t$  issued rules instructing the agent to choose  $a_t \in$

$\text{Argmax}_a \pi(a, \omega_t)$ . Constraints we impose on the communication, however, will usually make this impossible.

Our model of limited communication is built around a distance function  $\|\cdot\|$  defined on  $\Omega$ . A *rule*  $r$  is a quadruple  $r = (\omega, d, a, p)$ . This is interpreted as prescribing that action  $a$  should be chosen in period  $t$  if  $\|\omega_t - \omega\| < d$ . The extra parameter  $p$  is a real number that will be used to give some rules higher priority than others. We will sometimes write  $\omega(r)$ ,  $d(r)$ ,  $a(r)$ , and  $p(r)$  for the function giving the relevant component of the rule  $r$ .

A *rule book*  $R$  is a finite set of rules with a well defined precedence order:  $r, r' \in R \implies p(r) \neq p(r')$ . We assume that the agent follows any rule book given to him. If two or more rules conflict, the one with the highest priority is followed. If no rules apply the agent randomizes 50:50. Formally, write  $D(R)$  for the set of states covered by at least one rule,  $D(R) = \{\omega' \in \Omega \mid \|\omega' - \omega\| < d \text{ for some } (\omega, d, a, p) \in R\}$ . We assume that the agent's action choice is

$$a_t = \begin{cases} a & \text{if } (\omega, d, a, p) \in R_t, \|\omega_t - \omega\| < d, \text{ and } p = \max_{\{r \in R_t \mid \omega_t \in D(\{r\})\}} p(r) \\ \frac{1}{2} \cdot -1 + \frac{1}{2} \cdot 1 & \text{if } \omega_t \notin D(R_t). \end{cases}$$

Note that we have made assumptions directly on the agent's behavior rather than giving him a utility function. There are two possible interpretations of this. The first is that the agent as a robot who mechanically follows rules. For example, the agent could be a low skilled employee who does what he is told, but lacks the interest or sophistication to make inferences about when departing from the specified rules would be better. The second interpretation is that agents are short-run players with idiosyncratic private benefits that will result in their apparently choosing actions randomly if they have no financial incentives, and in which the feasible incentive contracts are exactly those that punish agents if they fail to follow a pre-specified rule. That is, other contracts are not enforceable.

The restriction to distance-based rules is by itself not much of a limitation on the principal: the first best could always be approximated (under appropriate regularity conditions) by a long finite list of rules. We will, however, also impose two other major restrictions on the set  $\mathfrak{R}_t$  of feasible rule books. First, we assume throughout the paper that only rules based on past observations are feasible and that at most one rule may be based on any

past observation:

$$\begin{aligned} r \in R_t \in \mathfrak{R}_t &\implies \omega(r) = \omega_{t'} \text{ for some } t' < t \\ r, r' \in R_t \in \mathfrak{R}_t &\implies \omega(r) \neq \omega(r'). \end{aligned}$$

Second, we will sometimes impose restrictions on how rules are changed over time. We consider three variants of our model

1. *No Overwriting.* The only feasible change to the rule book is to add a single rule that references the most recent state and applies to a previously uncovered domain, i.e.  $R_t \in \mathfrak{R}_t \implies R_t = R_{t-1} \cup \{r\}$  for some rule  $r$  with  $\omega(r) = \omega_{t-1}$  and  $D(\{r\}) \cap D(R_{t-1}) = \emptyset$ .
2. *Incremental Overwriting.* Once again the feasible changes are to add a single rule that references the most recent state. In this specification, however, the new rule is allowed to overlap with one or more previous rules and takes precedence if the rules conflict, i.e.  $R_t \in \mathfrak{R}_t \implies R_t = R_{t-1} \cup \{r\}$  for some rule  $r$  with  $\omega(r) = \omega_{t-1}$  and  $p(r) > p(r')$  for all  $r' \in R_{t-1}$ .
3. *Vast Overwriting.* In this specification there is no additional restriction on  $R_t$ . The principal can design an entirely new rule book in each period.

We have stated the model for an arbitrary state space  $\Omega$  and have in mind applications where  $\Omega$  is a complicated set (like the set of potential job applicants or potential entrepreneurs seeking funding) whose elements would be difficult to describe without reference to a previous example. Our analyses, however, will all take place in a simpler mathematical setting: we assume henceforth that  $\Omega = [0, 1]$  and define distances as if the state space was a circle,  $d(x, y) = \min\{|x - y|, 1 - |x - y|\}$ . Write  $f(\omega)$  for the payoff gain/loss from implementing action 1 at  $\omega$  rather than letting the Agent choose randomly,

$$f(\omega) = \pi(1, \omega) - \left( \frac{1}{2}\pi(1, \omega) + \frac{1}{2}\pi(-1, \omega) \right).$$

We assume that  $f(\omega)$  is continuous with  $f(0) = f(1)$ . We extend  $f$  to a periodic function on  $\mathbb{R}$  by setting  $f(x) = f(x) - f(\lfloor x \rfloor)$  to make some calculations easier. To rule out a

trivial case where rule-making is extremely easy or irrelevant, we also assume that  $f$  takes on both positive and negative values. Nothing changes if the payoffs are shifted up or down for each  $\omega$ , so we will also normalize payoffs by assuming that  $\pi(-1, \omega) + \pi(1, \omega) = 0$  for all  $\omega$ . With this assumption, the Principal's payoff is  $V = \sum_{t=1}^T \delta^t a_t f(\omega_t)$ .

## 2.2 An aside: A fully rational variant

Before proceeding to analyze our model let us illustrate one reason why we have chosen to make direct assumptions about how agents behave rather than specifying a “fully rational” model in which the agent maximizes a utility function. The reason is that our approach rules out schemes which allow the principal to communicate the first-best contingent plan, but which we would regard as unreasonable because they excessively exploit a dubious feature of traditional solution concepts – agents behave as if they know the equilibrium strategies of the other players.

For example, if we assume that the agent is rational with the same preferences as the principal, then the first-best can be obtained in a sequential equilibrium by having the principal effectively communicate the function  $f$ .

**Proposition 1** *Suppose that the function  $f$  has a finite number of zero crossings. Suppose the agent is fully rational and has the same preferences as the principal but doesn't know  $f$ . Then, there exists a sequential equilibrium in which the optimal action is taken in every period after the first with probability one.*

To see how this can be done, suppose that after  $\omega_1$  is observed by the principal and agent the principal issues a second-period rule  $(\omega_1, d, a, p)$  as follows. Rather than sending a rule that is to be interpreted literally, the principal uses the  $d$  part of the rule to encode several pieces of information. First, he sets the initial digits after the decimal point to be a binary string (zeroes and ones) which specifies the number of times  $f$  crosses zero. This is followed by a single digit of number 2, which is interpreted as a “stop code” indicating the next component. The second component is a single binary digit indicating which action to take on the first interval; it is followed by another 2. Finally, the location of each of the crossings is encoded as follows. The first digit of this component of the string is the

first digit of the first crossing, the second is the first digit of the second crossing, and so on for all  $n$  crossings. Then the second digit of the first crossing follows, and so on. In this manner, a single real number encodes everything about the function  $f$  that the agent needs to know to take the optimal action. Under the standard assumption that players play best responses to each others' strategies, it will be an equilibrium for the principal to send this message in the first period and for the agent to take the optimal action from the second period on.

We have not stated the proposition in the greatest possible level of generality. For example, one can also communicate the locations of a countable number of zero crossings by arranging them in a triangular array, e.g. first giving the first digit of one crossing, then the second digit of the first crossing and the first digit of a second, then the third digit of the first, the second digit of the second, and the first digit of the third, and so on. One could try to refine away these equilibria by ruling out equilibria which involve the agent processing an infinite string of bits, but there are many other coding schemes one could use. For example, the Principal could communicate any rule that can be described in finite space in English-language text by encoding the text in ASCII just as is done when sending an e-mail.

We think of this result not as a positive result, but rather as a cautionary note illustrating that the standard “rational” approach is unappealing for the problems we are trying to address. Specifically, the assumption of best-responding to equilibrium strategies embeds a degree of knowledge of the structure of the problem and of the meaning of language that does not fit with other aspects of our formulation. In the remainder of the paper we rule out schemes like the ones described above by assuming directly that agents interpret rules literally.

### **2.3 Another aside: Asymmetric analogies**

In this second aside we discuss the motivation for another assumption: our restriction to “symmetric” analogies. It is another conscious part of our modeling strategy – we see it as the simplest way to create the kind of accuracy vs. breadth tradeoff that will arise in any complex state space – and are happy with it even though we could have avoided the

obvious critiques via other means.

The symmetry restriction is that we have assumed the principal can only issue rules that apply when  $\|\omega_t - \omega\| < d$  rather than allowing rules that apply when  $\omega_t \in [\omega - d_1, \omega + d_2]$ . Many readers will no doubt quickly think of a two-step critique. First, asymmetric rules are used in practice, e.g. an admissions committee member could be told that students like STUDENT X should only be accepted if their technical background is as strong or stronger. Second, allowing asymmetric rules would have a dramatic effect in some situations. For example, when  $f$  is as pictured in Figure 3 and  $\omega_1 = \frac{3}{8}$ , a Principal restricted to asymmetric rules will be forced to define a highly inefficient rule, whereas a Principal who could explain that action 1 should be taken if the action is within  $\frac{3}{8}$  of  $\omega_1$  on the left or within  $\frac{1}{8}$  on the right would be able to perfectly describe where action 1 should be taken.

We agree with both steps, but still feel that our symmetry assumption is a very good one. Recall that our motivation for considering analogy-based rules is that the world is sufficiently complex so as to make it impossible to describe the sets of states in which each action is optimal. In such environments it is natural that all feasible rules will only crudely approximate the first-best. For example, if the set of states in which action 1 is optimal were the irregular shaped region in  $\mathbb{R}^2$  shown in Figure 2, then the best feasible rule would be inaccurate regardless of whether the agent could issue rules to apply on squares, or rectangles, and regardless of whether these had to be symmetrically centered on the point in question. We could have written this paper using a two-dimensional model without the symmetry restriction. Several of our propositions (including 2, 6, and 8) could be easily modified to apply in such an environment and examples could have been constructed to convey other points.

However, we feel that our simple one-dimensional model with the symmetry restriction captures the same breadth vs. accuracy tradeoffs and provides a neater way to convey our insights.

### 3 Two Periods

In this section we discuss the two-period version of our model. The two-period version is simple and brings out some basic insights.

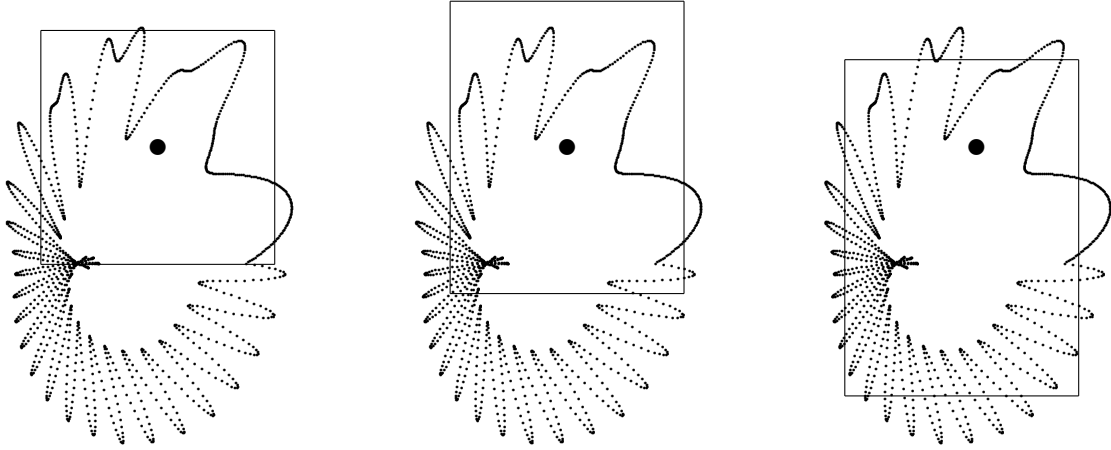


Figure 2: A higher-dimensional example

In the first period of our model, there is nothing for the principal to do and her expected payoff is always zero. In the second period, the principal can take advantage of the common reference point  $\omega_1$  to define a rule  $R_2$  that will result in a higher expected payoff. All three versions of our model are identical: the principal chooses a single rule  $(\omega_1, d, a)$ .

The next proposition notes that the principal is able to increase her payoff except in one special case and that rules are designed to have “excess breadth” in the sense that they are intentionally chosen to produce some incorrect decisions.

**Proposition 2** *Consider the two-period version of the models under any of the overwriting assumptions.*

1. *If  $f$  is not antisymmetric around  $\omega_1$  then all optimal rules have  $d^*(\omega_1) > 0$  and the principal’s expected second period payoff is positive,  $E(V_2|\omega_1) > 0$ .*
2. *Any interior optimal choice  $d^*(\omega_1)$  is such that  $f(\omega_1 - d^*(\omega_1)) = -f(\omega_1 + d^*(\omega_1))$ .*

Proof

The principal’s expected second period payoff with rule  $(\omega_1, d, a)$  is

$$E(V_2|\omega_1) = a \int_{\omega_1-d}^{\omega_1+d} f(x)dx.$$

This is a continuous function of  $a$  and  $d$  and the parameters are chosen from a compact set so it achieves its maximum. The maximum is zero only if  $\int_{\omega_1-d}^{\omega_1+d} f(x)dx = 0$  for all  $d$ , which implies that  $f$  is antisymmetric around  $\omega_1$ :  $f(\omega_1 - d) = -f(\omega_1 + d)$  for all  $d$ .

The second period payoff function is differentiable, so any interior optimum has

$$\left. \frac{d}{dd} \int_{\omega_1-d}^{\omega_1+d} f(x)dx \right|_{d=d^*} = 0$$

which implies  $f(x - d^*) = -f(x + d^*)$ .

QED

We will say that a rule  $r$  has *excess breadth* if there exists a state  $x$  for which  $|x - w(r)| < d(r)$  and  $a(r)f(x) < 0$ . The second part of Proposition 2 says that rules are expanded until the average value of the rule in the marginal cases,  $\omega_1 - d^*$  and  $\omega_1 + d^*$ , is zero. Unless both points are cases of indifference, this implies that the rule must be leading to incorrect decisions.

**Corollary 1** *If there does not exist a value  $d$  for which  $f(\omega - d) = f(\omega + d) = 0$ , then the optimal rule at  $\omega$  has excess breadth.*

Figure 3 provides a simple illustration. The left panel presents a simple payoff function. Action 1 is optimal whenever  $\omega \in (0, 1/2)$ . If the first period state is  $\omega_1 = 3/8$ , then the principal will instruct the agent to choose action 1 if a nearby state arises in period 2. She will not, however, choose a rule that is always correct by picking  $d = 1/8$ . At this breadth, she benefits from broadening the rule because the loss from choosing action 1 when  $\omega \approx \frac{1}{2}$  is much smaller than the gain from choosing action 1 when  $\omega \approx \frac{1}{4}$ . As noted in the proposition, the principal will, in fact, broaden the rule until it is completely worthless at the margin. In the illustrated case, the optimal choice is  $d^* = \frac{1}{4}$ . At this breadth, the gain from implementing action 1 at  $\omega_1 - d^* = \frac{1}{8}$  matches the loss from implementing action 1 at  $\omega_1 + d^* = \frac{5}{8}$ .

The right panel graphs the expected second-period payoffs as a function of the first period state. The principal is best-off if  $\omega_1 = \frac{1}{4}$  or  $\omega_1 = \frac{3}{4}$ . In these cases, the principal can issue a rule which is applicable half of the time and which always yields the correct decision when applicable. The payoff function turns out to be quadratic on the intervals around

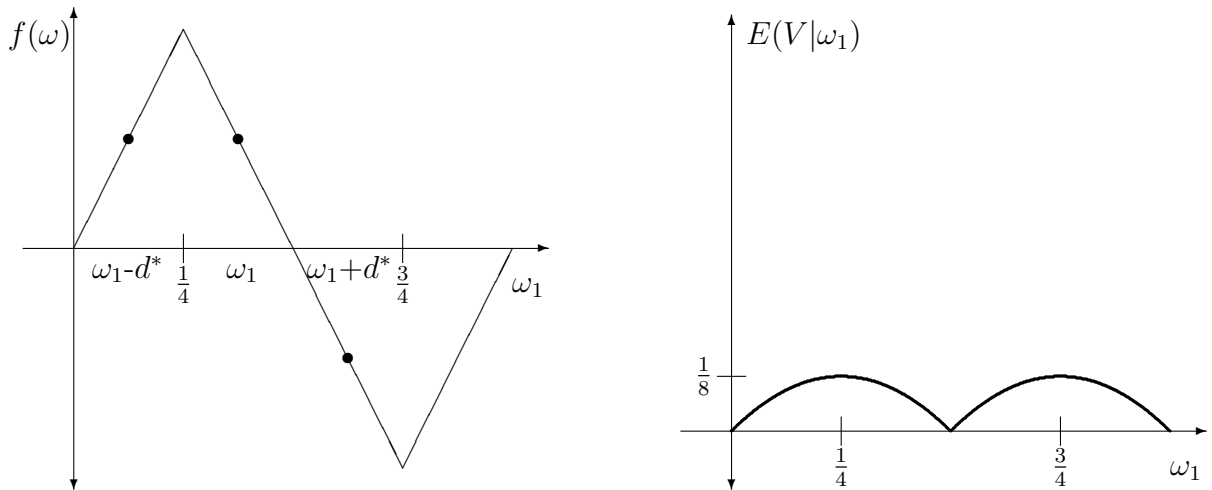


Figure 3: A two period example

the two maxima. When  $\omega_1$  is not equal to  $\frac{1}{4}$  or  $\frac{3}{4}$ , the principal must reduce the breadth of the rule and/or its accuracy and therefore ends up with a lower payoff. For the function in the example, it turns out that the principal entirely sacrifices accuracy: the optimal  $d^*$  is exactly one-quarter for almost all  $\omega_1$ . The only exceptions are the two states where the principal is worst off:  $\omega_1 = 0$  and  $\omega_1 = \frac{1}{2}$ . These states are particularly bad examples on which to base distance-based rules because they are right on the boundary between the regions where the two actions are optimal. In these states the principal's second period expected payoff is zero for any choice of  $d$  and  $a$ .

### Observations

1. The fact that the first-period draw of  $\omega_1$  primarily effects the accuracy of the chosen rule and not its breadth is somewhat more general than the above example. Suppose that  $f$  is uniquely maximized at  $y$ , uniquely minimized at  $y + \frac{1}{2}$ , that  $f$  is strictly monotonic between the minimum and maximum, and  $f$  is symmetric both in the sense that  $f(y - z) = f(y + z)$  for all  $z$  and in the sense that  $f(x + \frac{1}{2}) = -f(x)$  for all  $x$ . Then, the principal's optimal choice will be  $d^* = \frac{1}{4}$  for almost all  $\omega_1$ . An easy argument for this is that the first-order conditions imply that  $f(\omega_1 - d^*) = -f(\omega_1 + d^*)$ . Symmetry gives  $f(\omega_1 - d^*) = f(\omega_1 + d^* - \frac{1}{2})$ .<sup>2</sup> The function  $f$  takes on

<sup>2</sup>For some values of  $\omega_1$  and  $d$  the term on the left will need to be  $f(\omega_1 - d^* + 1)$  and/or the term on

each value only twice. Hence we either have that the two arguments are the same,  $\omega_1 - d^* = \omega + d^* - \frac{1}{2}$ , or that the arguments are equidistant from the peak on opposite sides,  $y - (\omega_1 - d^*) = \omega_1 + d^* - \frac{1}{2} - y$ . The former gives  $d^* = \frac{1}{4}$ . The latter implies that  $\omega_1 = y + \frac{1}{4}$ , a non-generic case.

2. The principal would typically be able to achieve higher payoffs if we allowed her to issue multiple rules based on the same state. For example, given the function  $f$  pictured in Figure 1 she could achieve the first-best payoff in the second period if the first period state was  $\omega_1 = \frac{1}{4}$  by issuing two rules: a lower-priority rule telling the agent to take action -1 if  $w_2$  is within distance infinity of  $\omega_1$ , and a higher priority rule telling the agent to take action 1 if  $w_2$  is within distance  $\frac{1}{4}$  of  $\omega_1$ . In this case, the rules are perhaps not so unreasonable: they are like telling the agent to do action 1 in some cases and action -1 in all others. In other cases, however, we felt that such overlapping rules seemed to be less reasonable, e.g. the principal could dictate  $N$  different actions on concentric rings around  $w_1$  by defining a first rule that applied at distances less than  $d_1$ , a second higher-priority rule that applied at distances less than  $d_2 < d_1$ , a third even higher priority rule that that applied at distances less than  $d_3$ , and so on. Such rules may be an interesting additional category of rules to study, but they go against our motivation of studying principals who must rely on simple analogy-based rules to cope with complexity, so we decided to limit our current paper to just three variants.

## 4 Three Periods

In this section, we discuss the three period version of our model. A primary observation is that the option value of being able to define superior rules in the future reduces some of the excess breadth we saw in the two-period case.

Write  $r_2^m(\omega_1)$  for the myopic optimal rule at  $t = 2$ . This can be defined either as the optimal choice in the two-period model described in the previous section or as the optimal choice in the three-period model for a principal with discount factor  $\delta = 0$ . Let 

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the right-hand side will need to be  $f(\omega_1 + d^* - \frac{3}{2})$  or  $f(\omega + d^* + \frac{1}{2})$  ensure that the values are in  $[0, 1]$ .

$d_2^m = d(r_2^m(\omega_1))$  be the breadth of the myopic optimal rule. We say that this rule has *excess breadth at the margin* if  $\min(f(\omega_1 - d_2^m), f(\omega_1 + d_2^m)) < 0$ . Write  $r_2^*(\omega_1)$  for the optimal rule at  $t = 2$  for an agent with discount factor  $\delta > 0$ .<sup>3</sup>

## 4.1 No overwriting

Our first result on the no-overwriting model highlights that option value considerations lead the principal to define narrower rules in the three-period model than would be used in the two-period model.

**Proposition 3** *Suppose no overwriting is allowed. Let  $f$  be differentiable and suppose that the myopic optimal rule  $r_2^m(\omega_1)$  is a unique interior optimum and has excess breadth at the margin. Then, the optimal rule at  $t = 2$  in the three-period model has  $d(r_2^*(\omega_1)) < d_2^m$ .*

### Proof

We show that the expected payoff from any rule  $r'_2, r'_3(\omega_2)$  with  $d'_2 \equiv d(r'_2) \geq d_2^m$  is strictly lower than the expected payoff that is obtained from a strategy we'll specify that includes setting  $r_2 = (\omega_1, d_2^m - \epsilon, a(r_2^m(\omega_1)))$  for a small  $\epsilon > 0$ .

First, consider any strategy  $r'_2, r'_3(\omega_2)$  with  $d(r'_2) > d_2^m$ . We show that such strategies cannot be optimal by showing that the expected payoff is improved by switching to the strategy  $r_2^m, r'_3(\omega_2)$ . To see this, note first that  $r_2^m$  provides a strictly greater expected payoff in the second period. In the third period, the expected payoff conditional on  $\omega_3 \in [\omega_1 - d(r'_2), \omega_1 + d(r'_2)]$  is again greater under  $(r_2^m, r'_3(\omega_2))$  by the optimality of  $r_2^m$ . The third period expected payoff conditional on  $\omega_3 \notin [\omega_1 - d(r'_2), \omega_1 + d(r'_2)]$  is identical under the two rules.

Next, consider any strategy  $r'_2, r'_3(\omega_2)$  with  $d(r'_2) = d_2^m$ . We show that the expected payoff can be improved by switching to a strategy with a slightly smaller value of  $d_2$ . To define this strategy, note first that the assumption that  $d_2^m$  is regular and has excess breadth at the margin implies that  $f(\omega_1 \pm d_2^m) \neq 0$ . Assume WLOG that  $f(\omega_1 + d_2^m) > 0$  and  $f(\omega_1 - d_2^m) < 0$ . Hence, we can choose  $\eta > 0$  so that  $f(\omega + x) > f(\omega + d_2^m)/2$

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<sup>3</sup>This is obviously a function of  $\delta$ . We omit the dependence from the notation where it will not cause confusion.

and  $f(\omega - x) < f(\omega - d_2^m)/2$  for all  $x \in [d_2^m, d_2^m + 3\eta]$ . For  $0 < \epsilon < \eta$  define  $r_2(\epsilon) = (\omega_1, d_2^m - \epsilon, a(r_2^m(\omega_1)))$  and

$$r_3(\omega_2; \epsilon) = \begin{cases} r_3'(\omega_2) & \text{if } \omega_2 \notin [\omega_1 - d_2^m - \eta, \omega_1 + d_2^m + \eta] \\ (\omega_2, \|\omega_2 - (\omega_1 + d_2^m - \epsilon)\|, 1) & \text{if } \omega_2 \in [\omega_1 + d_2^m - \epsilon, \omega_1 + d_2^m + \eta] \\ (\omega_2, \|\omega_2 - (\omega_1 - d_2^m + \epsilon)\|, -1) & \text{if } \omega_2 \in [\omega_1 - d_2^m - \eta, \omega_1 - d_2^m + \epsilon] \end{cases}$$

In words, these strategies consist of narrowing the breadth of rule  $r_2$  by  $\epsilon$  and then taking advantage of the more narrow definition to choose an  $\epsilon$  broader rule than would have been possible in these cases when the realization of  $\omega_2$  within  $\eta$  of the boundary of the initial rule. We show that  $r_2(\epsilon), r_3(\omega_2; \epsilon)$  gives a higher payoff than  $r_2', r_3'(\omega_2)$  when  $\epsilon$  is small.

To see this note first that the disadvantage of  $r_2(\epsilon)$  in the second period is  $O(\epsilon^2)$  because payoffs are lower if  $\omega_2 \in I \equiv [\omega_1 - d_2^m, \omega_2 - d_2^m + \epsilon] \cup [\omega_1 + d_2^m - \epsilon, \omega_2 + d_2^m]$ . In the third period payoffs differ in two cases: if  $\omega_3 \in I$ ; and if  $\omega_2$  is in the interval where the third-period rule is different and  $\omega_3 \notin I$ . The former difference again gives a loss that is  $O(\epsilon^2)$ . The latter case gives a larger advantage. In particular, if  $\omega_2 \in [\omega_1 - d_2^m - \eta, \omega_1 - d_2^m + \epsilon]$ , then the correct decision is made whenever  $\omega_3 \in [\omega_2 - z - \epsilon, \omega_2 - z]$  for  $z = \|\omega_2 - (\omega_1 - d_2^m)\|$ , whereas previously no rule was defined in this interval. The gain from this in expected value terms is at least  $\epsilon f(\omega_1 - d_2^m)/2$ . There is a similar  $O(\epsilon)$  gain from getting the correct decision when  $\omega_2 \in [\omega_1 + d_2^m - \epsilon, \omega_1 + d_2^m + \eta]$  and  $\omega_3$  is in  $[\omega_2 + z, \omega_2 + z + \epsilon]$  for  $z = \|\omega_2 - (\omega_1 + d_2^m)\|$ . Hence, for sufficiently small  $\epsilon$  the strategy  $r_2(\epsilon), r_3(\omega_2; \epsilon)$  gives a higher payoff.

QED

Figure 4 provides an illustration. The left panel graphs the the breadth of the myopic optimal rule,  $d(r_2^m(\omega_1))$ , and the breadth of the optimal second-period rule in the three-period model,  $d(r_2^*(\omega_1))$  for same function  $f$  pictured in Figure 3.<sup>4</sup> Recall that the myopic optimal rule had a distance of  $\frac{1}{4}$  for almost all  $\omega_1$ . This is shown as a dotted line in the graph. In the three period model, this breadth is never optimal By Proposition 3 the optimal second-period rule is narrower. The solid line in the figure gives this breadth. Note that sometimes the optimal rule has zero breadth. This is the case near  $\omega_1 = 0$ ,  $\omega_1 = \frac{1}{2}$  and  $\omega_1 = 1$ . For other values of  $\omega_1$  the optimal rule has breadth close to  $\frac{1}{4}$ .

<sup>4</sup>The discount factor of  $\delta = 1$  was used for these graphs.

The right panel graphs the expected payoff as a function of  $\omega_1$ . In the region where the optimal rule has positive breadth, the payoff function again declines smoothly away from maxima at  $\omega_1 = \frac{1}{4}$  and  $\omega_1 = \frac{3}{4}$ .

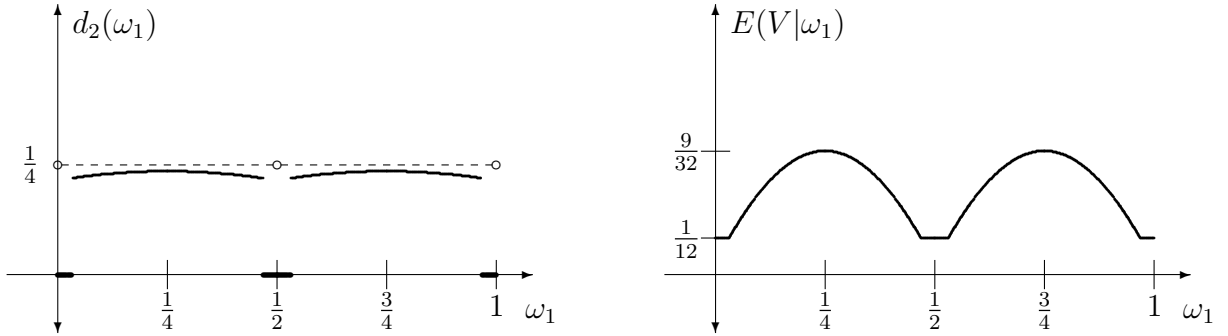


Figure 4: A three period example

An interesting observation from the graph is that the optimal second-period rule in the three-period no overwriting model can have *insufficient breadth at the margin*:  $d_2^* < \frac{1}{2}$  and  $f(\omega_1 - d_2^*)$  and  $f(\omega_1 + d_2^*)$  are both of the same sign as  $a_2^*$ .

**Proposition 4** *There exist payoff functions for the three-period no-overwriting model for which the the optimal second-period rule has insufficient breadth at the margin for a positive measure of values of  $\omega_1$ .*

The left panel of Figure 4 illustrates that the payoff function given above provides the necessary example. One case in which it is intuitive that the optimal rule will have insufficient breadth is when when  $\omega_1$  close to zero, one-half, and one. Here, the optimal breadth is zero, which will satisfy the insufficient breadth at the margin condition. The optimal breadth is zero because the option value to keeping rules undefined outweighs the potential short-run benefit to making a rule. For example, suppose  $\omega_1 = \frac{1}{2} + \epsilon$ . In this case, a second-period gain of approximately  $\epsilon$  could be obtained by making a rule that action -1 should be chosen if  $\omega \in [\frac{1}{4} + \epsilon, \frac{3}{4} + \epsilon]$ , or a smaller gain of  $\frac{1}{2}\epsilon^2$  could be obtained by defining a much narrower rule that action -1 should be chosen on  $[\frac{1}{2}, \frac{1}{2} + 2\epsilon]$ . However, in each case, such a rule can prevent a much better rule from being defined in the third

period. For example, if  $\omega_2 = \frac{3}{8}$ , the former would prevent us from defining any rule at all, and the latter would force us to limit the domain of the rule to  $[\frac{1}{4}, \frac{1}{2}]$  instead of  $[\frac{1}{8}, \frac{5}{8}]$ .

A second noteworthy situation in which we get insufficient breadth at the margin is when  $\omega_1 = \frac{1}{4}$ . In this situation one's initial thought might be that the principal has received a fortunate draw and will choose  $d = \frac{1}{4}$  to define the exactly optimal rule on half of the states. This exactly optimal rule, however, is still of zero value at the margin. Under the no overwriting assumption the marginal breadth has an opportunity cost. For example, if the second period state is  $\omega_2 = \frac{5}{8}$  the principal will be able to dictate that action -1 be taken when  $\omega_3 \in [\frac{1}{2}, \frac{3}{4}]$ , but will be unable to promulgate a broader rule covering  $[\frac{1}{2} - \epsilon, \frac{3}{4} + \epsilon]$ .

We omit a formal proof of Proposition 4 because it is unexciting and the graph serves as a numerical proof.

## 4.2 Incremental overwriting

The above discussion has focused on the no-overwriting version of our model. In the incremental overwriting model, some of the constraints which we had mentioned in explaining why a principal might intentionally make a rule excessively narrow no longer exist. For example, the principal can overwrite the interval  $[\frac{1}{2} - \sigma, \frac{1}{2}]$  at  $t = 3$  if the draw of  $\omega_2$  makes this attractive. The incremental overwriting model does have some constraints, however, and these can still provide an incentive to choose a narrower second-period rule than would be chosen by a myopic agent.

**Proposition 5** *In the three period incremental overwriting model there exist payoff functions  $f$  and states  $\omega_1$  for which  $d(r_2^*(\omega_1)) < d_2^m$ .*

### Proof

We prove this by providing an example. Suppose  $f(x) = 1$  if  $x \in [\epsilon, \frac{1}{3} - \epsilon]$  and  $f(x) = -1$  otherwise.<sup>5</sup> For  $\omega_1 = \frac{1}{6}$  the myopically optimal rule is to take action -1 everywhere: this gives a payoff of  $\frac{1}{3} + 4\epsilon$ , whereas the more obvious narrow rule dictating that action 1 be taken on  $[\epsilon, \frac{1}{3} - \epsilon]$  gives a payoff of  $\frac{1}{3} - 2\epsilon$ .

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<sup>5</sup>The example uses a discontinuous  $f$  to make the computations easy. A continuous example could be obtained by choosing a nearby continuous function.

In the three-period incremental overwriting model, however, the narrower rule provides a strictly greater expected payoff if  $\epsilon$  is small and  $\delta$  is close to one. To see this, note that the narrower rule can be improved in the third period whenever  $\omega_2 \notin [\epsilon, \frac{1}{3} - \epsilon]$ . When such  $\omega_2$  arise, the optimal incremental rule is to prescribe action -1 on the largest interval around  $\omega_2$  contained in  $[\frac{1}{3} - \epsilon, 1 + \epsilon]$ . On average this interval has width  $\frac{1}{3} + \epsilon$ , so the expected gain is  $(\frac{2}{3} + 2\epsilon)(\frac{1}{3} + \epsilon) \approx \frac{2}{9}$ . The broad rule can be improved only if  $\omega_2 \in [\epsilon, \frac{1}{3} - \epsilon]$ . Again, the best way to improve the rule is to define  $r_3$  to extend to the nearest edge of the interval  $[\epsilon, \frac{1}{3} - \epsilon]$ . There is a payoff gain of two whenever  $\omega_3$  is such that the correction is effective, but the expected gain is still just  $2(\frac{1}{3} - 2\epsilon)(\frac{1}{6} - \epsilon) \approx \frac{1}{9}$ . The nontrivial difference in third period expected payoffs will easily outweigh the  $\epsilon$  order second-period differences if  $\delta$  is close to one and  $\epsilon$  is not too large.

QED

An intuition for why the narrower rule is preferable in the example in the above proof is that a constraint imposed by the incremental overwriting model is that one cannot go back at  $t = 3$  and redefine the rule involving  $\omega_1$ . Hence, there is a benefit to immediately employing  $\omega_1$  in its best long-run role. In this case, the natural use for  $\omega_1 = \frac{1}{6}$  is as the leading example of of the set of situations where action 1 is optimal.

### 4.3 Vast overwriting

In the vast overwriting model there is no incentive to leave rules undefined. The long-run optimal strategy is always to define each period's rulebook to maximize the payoff in that period. Hence  $r_2^* = r_2^m$ .

## 5 Infinite Horizon

In an infinite horizon model, rule books become more complete over time. In the vast overwriting model this occurs because the principal eventually has access to a great variety of examples on which rules can be based. In the no-overwriting and incremental-overwriting models this occurs because rules never go away once promulgated, and the principal must issue rules with positive probability whenever the rule book is incomplete – there are always states on which a useful rule can be based and the only reason not to introduce

a rule when such opportunities arise is if there is positive probability that a situation in which the principal will define an even better rule will arrive. In this section, we first present a convergence result for general discount factors showing that rule books converge over time in the sense that the expected payoff a rule book produces in each period almost surely converges to a limiting value. Of course, history matters: for some realizations of  $\omega_0, \omega_1, \dots, \dots$  the limiting rule book will be highly efficient and for others it may be very inefficient. It is natural to expect that a patient principal ( $\delta$  close to 1) will earn a high long-run payoff by building up a rule book that prescribes nearly optimal actions. The second set of results we present illustrate the sense in which this is true.

## 5.1 Limiting Payoffs

In the no-overwriting and vast-overwriting versions of the model it is easy to show the principal's payoff converges to a well-defined limiting payoff in the long run.

**Proposition 6** *Consider the no-overwriting or vast-overwriting versions of the model. Write  $\pi_t \equiv E(\pi(a_t, \omega_t | R_t(\omega_0, \dots, \omega_{t-1})))$  for the expected payoff in period  $t$  given the rule book which the principal has defined in equilibrium. Then, there exists a limiting random variable  $\pi_\infty$  such that  $\pi_t$  converges to  $\pi_\infty$  almost surely and in distribution.*

### Proof

In both versions of the model we have  $E(\pi_t | \pi_{t-1}) \geq \pi_{t-1}$ . In the vast overwriting case this is because it is always possible to define a weakly better rule book with a superset of previous realizations. In the no-overwriting case this is because the rule that the principal defined in period  $t - 1$  must have been better than defining a rule dictating that the opposite action be taken on the same domain. The process  $\pi = (\pi_t : t \geq 0)$  is adapted and  $E(|\pi_t|)$  is finite for all  $t$ , so  $\pi$  is a submartingale. Hence, Doob's Forward Convergence Theorem (Williams 1991, p.109) applies and guarantees the existence of a  $\pi_\infty$  with the stated properties.

QED

Informally, the “almost surely” part of the statement of the proposition is saying that the rule book converges over time for almost any realization of the sequence  $\omega_0, \omega_1, \omega_2, \dots$ . The “in distribution” part of the conclusion tells us that the distribution of the random variable

$\pi_\infty$  describes how often we end up with a limiting rule book that is very efficient/moderately efficient/highly inefficient in the long run.

Figure 4 plots the density of  $\pi_\infty$  for the example we have used throughout under the assumption that the principal acts myopically ( $\delta = 0$ ). Note that there is a significant amount of mass close to  $1/4$ , but also in the center of the support. This highlights the fact that the long-run payoff depends on the sequence of realization of  $\omega$ . That is, there is path dependence. A series of early “good” realizations—in the sense that they allow effective rules to be created—have a persistent effect and *vice versa*. In this example, highly efficient limit rule books are much more common than highly inefficient limit rule books. The limit rule book will end up being highly efficient whenever the initial state is close to  $1/4$  or  $3/4$ . In contrast, a nearly completely inefficient rule book only arises if the principal receives multiple unlucky draws. With a patient principal highly inefficient outcomes should become even more unlikely – there should be zero probability of developing a limit rule book with a payoff in some neighborhood of zero because the principal will refuse to define any rule in period  $t$  if the realization of  $\omega_t$  is sufficiently unfortunate.

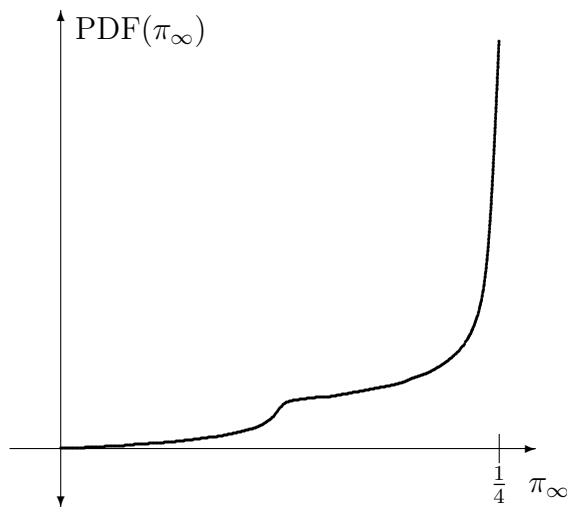


Figure 5: Distribution of long-run payoff  $\pi_\infty$  with myopic play

## 5.2 Vast overwriting with $\delta$ close to 1

Developing rules is easiest in the vast-overwriting version of our model. Here, it is fairly easy to see that a fully optimal rule book is not only achieved asymptotically, but develops in finite time if the function  $f$  is well behaved.

The first-best rulebook would specify that  $a_t = a^{FB}(\omega_t) \equiv \text{sign}(f(\omega_t))$ . Write  $V^{FB} = \int_0^1 |f(x)| dx$  for the per-period payoff that this rule would give. This is an upper bound on the expected payoff that a rational player can achieve in any period.

**Proposition 7** *Consider the infinite horizon vast-overwriting version of our model with any discount factor  $\delta > 0$ . Suppose that  $f(x)$  crosses zero a finite number of times. Then, with probability one there exists a  $T$  such that the action  $a^{FB}(\omega_t)$  is chosen in period  $t$  for all  $t \geq T$ . As  $\delta \rightarrow 1$  the principal's average per-period payoff converges to  $V^{FB}$ .*

### Proof

The second conclusion follows directly from the first. To see the first, suppose that  $f(x)$  crosses zero  $n - 1$  times and let  $0 = x_0 < x_1 < \dots < x_n = 1$  be such that  $f(x)$  is always nonnegative or always nonpositive on each interval  $S_j = [x_{j-1}, x_j]$ . Write  $a^j$  for the optimal action on  $S_j$ . Let  $S_{j1} = [x_{j-1} + \frac{1}{4}(x_j - x_{j-1}), x_{j-1} + \frac{1}{2}(x_j - x_{j-1})]$  and  $S_{j2} = [x_j - \frac{1}{2}(x_j - x_{j-1}), x_j - \frac{1}{4}(x_j - x_{j-1})]$ . With probability one there exists a  $T$  such that  $\{\omega_1, \omega_2, \dots, \omega_T\} \cap S_{jk} \neq \emptyset$  for all  $j = 1, 2, \dots, n$  and for  $k = 1, 2$ . For any  $t > T$  we can define  $\tau(j, k)$  so that  $\omega_{\tau(j, k)} \in S_{jk}$ . Define a rulebook for period  $t$  by

$$R_t = \bigcup_{j=1}^n \{(\omega_{\tau(j,1)}, \omega_{\tau(j,1)} - x_{j-1}, a^j, j), (\omega_{\tau(j,2)}, x_j - \omega_{\tau(j,2)}, a^j, -j)\}$$

Note that one or more of the rules applies to each  $\omega \in [0, 1]$  and that for any  $\omega \in S_j$  any rule that applies to  $\omega$  specifies action  $a^j$ .<sup>6</sup> Hence,  $R_t$  will result in action  $a^{FB}(\omega_t)$  being chosen.

QED

Remarks

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<sup>6</sup>When two rules apply they give the same action so the priorities we specified were unimportant.

1. The assumption that  $f(x)$  crosses zero a finite number of times is necessary for the first part of the result. If  $f(x)$  has an infinite number of crossings then no finite rule can specify the optimal action in between each of the crossings. Hence, the upper bound  $V^{FB}$  cannot be achieved in any period.
2. The rulebook used in the proof employs rules with overlapping domains that specify the same action on the overlap. Some such overlapping is necessary to achieve the first-best in finite time. If compatible rules are not allowed to overlap, then on each interval  $S_j$  we will be able to define rules that extend exactly to the right endpoint or exactly to the left endpoint, but not both (unless the  $\omega$ 's satisfy a relation that is satisfied with probability zero). We need rules to end exactly at  $n + 1$  endpoints to achieve the first best. Hence, the first-best is not achievable.

### 5.3 No overwriting and incremental overwriting with $\delta$ close to 1

The no overwriting and incremental overwriting versions of our model impose additional constraints on the principal and will reduce her payoffs. Nonetheless, it is fairly easy to show that payoffs still must approximate the first-best when the principal is patient.

**Proposition 8** *Suppose that  $f(x)$  crosses zero a finite number of times. In the no overwriting and incremental overwriting models the principal's average per-period payoff converges to  $V^{FB}$  in the limit as  $\delta \rightarrow 1$ .*

#### Proof

A suboptimal strategy for the principal which is feasible in the incremental overwriting model would be to define no rule except in the  $2n$  periods in which the current state falls into one of the subintervals  $S_{jk}$  defined in the proof of Proposition 7 for the first time, and in those to define the rule exactly as we do in that proof. If the principal follows this strategy she will achieve a payoff  $V^{FB}$  in all periods from some period  $T$  on. Hence, her average payoff will be something that approaches  $V^{FB}$  as  $\delta \rightarrow 1$ . Her optimal strategy yields a higher expected payoff, and hence must also approach  $V^{FB}$ .

The above strategy is not feasible in the no overwriting model because the principal is restricted from issuing overlapping rules even when they disagree. More complicated strategies can be used, however, to obtain a payoff that approximates the first best. One way to do this is to define no rule unless the state is within  $\epsilon$  of the center of one of the  $S_j$ . There may be a long wait until such states occur, but once they have occurred in each  $S_j$  the expected payoff in all future periods is at least  $1 - 2n\epsilon$ . This similarly puts a lower bound on the payoff that a rational player can receive.

QED

Remark

1. The assumption that  $f(x)$  has a finite number of crossings is convenient for this proof, but is not necessary. If  $f(x)$  crosses zero a countable number of times, we can implement a similar strategy in each subinterval. Not all subintervals will be covered, but each half of a subinterval of width  $w$  is covered with probability  $(1 - \frac{w}{2})^{t-1}$  in period  $t$ . Hence, the expected payoff again converges to the first-best.

A final result of this section is that when no overwriting is allowed the first-best is not achieved in finite time. As in the situation described in the second remark after Proposition 7, this is a mechanical consequence of the principal's limited ability to define rules. This time we do give a formal statement and a proof.

**Proposition 9** *Consider the no overwriting version of our model. Assume that there is no nontrivial subinterval  $(a, b)$  on which  $f$  is almost everywhere equal to zero. Then,  $\text{Prob}\{E(V_{t+1}|\omega_1, \dots, \omega_t) = V^{FB}\} = 0$  for every  $t$ .*

Proof

Let  $\Omega^\infty$  be the set of all sequences  $\{\omega_t\}$ . We show the result by partitioning  $\Omega^\infty$  into two disjoint subsets  $\Omega^\infty = \Omega_1 \cup \Omega_2$  and showing that the result holds for almost all  $\{\omega_t\}$  in each  $\Omega_i$  provided that  $\Omega_i$  is not of measure zero.

The division is simple: we write  $d^0$  for the width of the first nontrivial rule issued by the principal, i.e.  $d^0 = d(r_s)$  where  $s$  is such that  $d(r_s) > 0$  and  $d(r_t) = 0$  for all  $t < s$  and set  $\Omega_1$  to be the set of all  $\{\omega_t\}$  with  $d^0 \geq \frac{1}{2}$ .

Case 1:  $\{\omega_t\} \in \Omega_1$

In this case, the agent chooses at random in the first  $s - 1$  periods and chooses a state-independent action in all future periods. Hence, we have  $E(V_t|\omega_1, \dots, \omega_{t-1}) = 0$  for all  $t < s$  and  $E(V_t|\omega_1, \dots, \omega_{t-1}) \leq |\int f(x)dx| < \int |f(x)|dx = V^{FB}$  for all  $t > s$ , with the strict inequality following from our assumption that  $f$  is continuous and takes on positive and negative values.

Case 2:  $\{\omega_t\} \in \Omega_2$

It suffices to show that with probability one there is a nonempty interval that is not covered by any rule. This follows easily by induction. In period  $s$  there is an interval of width  $1 - 2d^0$  that is uncovered. If in any period  $t$ , some interval of width  $w$  with  $0 < w < 1$  is uncovered, then a subinterval remains uncovered in period  $t + 1$  unless  $\omega_t$  lies exactly in the middle of the uncovered interval from period  $t$ . This occurs with probability one.

QED

## 6 Delegation Structures

So far our environment has been sufficiently simple that there has been a single possible delegation structure for the authority relationship/firm: a single principal, a single agent and a single task. We now consider a richer environment where the principal has a single unit of time, but may do two things: perform a task herself, or communicate about it to an agent. We assume that performing a task is more time consuming than communicating about it. In particular, we assume that performing a task takes one unit of time, but communicating to an agent takes only  $1/2$  a unit of time. The principal has two agents available.

More concretely, suppose there are two tasks  $i = 1, 2$ . The principal's payoff on task  $i$  is  $\pi_i(a_i, \omega_i)$ , where  $a_i$  is the action taken on task  $i$  and  $\omega_i$  is a task-specific state of nature. As before, the principal's payoff in the full game is the discounted sum of his per period payoffs:  $V = \sum_{t=1}^T \delta^t \pi_1(a_{1t}, \omega_{1t}) + \pi_2(a_{2t}, \omega_{2t})$ . In each period, Principal has two options. First, he can communicate rules to both agents at the start of the period. They will then carry out these instructions as in our standard model. We call this "full delegation". Second, the

Principal can instead carry out one of the actions herself. If she does this, we assume that she does not have sufficient time to communicate with the other agent so that agent carries out the most recently communicated rule book. We assume the Principal does observe both  $\omega_1$  and  $\omega_2$  and can base rules on these in the future. We call this “partial delegation”.

In some situations it would be appropriate to assume that delegation structures are time invariant. In this case we have:

**Proposition 10** *Consider any of the overwriting assumptions, suppose that  $f(x)$  crosses zero a finite number of times, and that delegation strategies are time invariant. Then if  $\delta$  is sufficiently large then full delegation is optimal.*

Proof

From Propositions 7 and 8 we know that as  $\delta \rightarrow 1$  the principal’s average per-period payoff converges to  $V^{FB}$ . Partial delegation cannot achieve this since there is no communication on one of the tasks and the expected payoff on that task is zero. Even though the first best is obtained on the other task, the sum of the payoffs on the two tasks will be worse than a payoff that approaches the first-best on both as  $\delta \rightarrow 1$ .

QED

In other situations it may be reasonable to assume that firms can change their delegation structure as the firm matures. In this case, our model becomes a theory of organizational change. In the first period, the Principal must practice partial delegation. Full delegation will provide no benefits because there is no history on which to base rules, whereas with partial delegation she can achieve the first-best payoff on one of the two tasks. In the vast overwriting case, at least, the Principal must switch at some point to full delegation: as history accumulates it will eventually become possible to achieve a superior payoff by delegating both tasks. (Switching directly to partially delegating the other action is inferior because before any rule has been communicated it provides a payoff that is no better than the payoff that was achievable at the start of the game.) The long-run organization structure is indeterminate. If the first-best is achievable when the Principal first switches to full delegation then there is obviously no benefit from further switches in organizational form. But if the Principal can communicate the first-best rule on exactly one task at this

time, then she strictly prefers to use partial delegation at least for some period of time after this.

Other variants of this model could be considered. For example, one could assume that the Principal and Agent do not accumulate shared experiences when the Principal is working on the other task. Principals in such models will face a tradeoff: do they take an optimal action now or invest in developing shared experience. The dependence which the basic model exhibited can be magnified in such settings: there is both a direct effect of history on payoffs and an additional indirect effect whereby a fortunate history allows the Principal to adopt a delegation structure that is more efficient in the long run. We leave the exploration of such models to future work.

## 7 Endogenous Categories

As we mentioned in the introduction, our model suggests a mechanism via which categories are formed in models of category based thinking (such as Mullainathan (2002) or Fryer and Jackson (2007)). Roughly speaking, these models depart from the Bayesian framework by grouping together multiple states into a single category. In Fryer and Jackson the decision maker allocates experiences to a finite and limited number of categories—and they focus on the optimal way to categorize (given the number categories available). They show that experiences which occur less frequently are (optimally) placed in categories which are more heterogeneous. Mullainathan (2002) considers decision makers who have coarser beliefs than pure Bayesians, since they have a set of categories which are a partition of the space of posteriors and choose just one category given data which they observe. They then forecast using the probability distribution for the category.

An important question which these models leave open is where exactly these categories come from. In Mullainathan (2002) they are completely exogenous. In Fryer and Jackson (2007) they are formed optimally, although the number of categories is exogenous. Our model provides one way to think about how categories are formed. In our setting we think of the categories as being the sets  $\{\omega \mid \|\omega - \omega_t\| < d_t\}$  on which individual rules are defined. Thus in an  $N$  period model if action 1 is taken on several disjoint set we think of

them as being different categories<sup>7</sup>. The analog of perfect categorization is that all states are covered by a rule (with highest precedence) which maximizes the principal’s payoff. Imperfect categorization comes from one of two sources: (i) rules which do not maximize the principal’s payoff in a given state or, (ii) states which are not covered by a rule.

Fryer and Jackson (2007) offer a labor market example with high and low ability workers who may be black or white. Thus there are four types: high-black, low-black, high-white, low-white. They assume that the respective population shares of the types are: 5%, 5%, 45%, 45%. In the context of their model they show that if there are only three categories then high and low ability blacks are categorized together, whereas high-white and low-white each have their own category.

In the context of our model, suppose that action 1 is the decision to hire a worker and action -1 is the decision not to hire. The left panel of figure 4 depicts an arrangement of workers which (from left to right) is: high-white, low-white, low-black, high-black. The dotted vertical line represents the division between whites and blacks. Now suppose that there are two periods and the first period realization of the state is  $\omega_1$ . The optimal rule categorizes all workers in the band between the first and third dots together and the rule is that any worker in that category should be hired. In a sense this rule is quite “fair”, though not perfectly so. Most high ability blacks are categorized as to-be-hired, though not all, and a few low ability whites are also put in that category.

In the second panel of figure 4 outcomes can be much less fair. The difference between the two panels is the frequency of the function. This captures the idea that high and low ability types are harder to distinguish than they are in the left panel—they are less naturally clumped together. Not only can this lead to a less effective rule being developed, it can lead to a less fair one. If the realization of the state is  $\omega_1$  then the optimal rule categorizes all workers between the first and third dots as not-to-be-hired. More than half the high-ability blacks are so categorized, and this comes from the principal’s desire to expand the rule on the left of  $\omega_1$  so as to prevent many low-whites from being hired. The key driver of the unfairness is that whites are relatively more populous than blacks. This is a similar underlying cause as in Fryer and Jackson, though the models are obviously different. In a

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<sup>7</sup>In the case of overlap a state can belong to multiple categories.

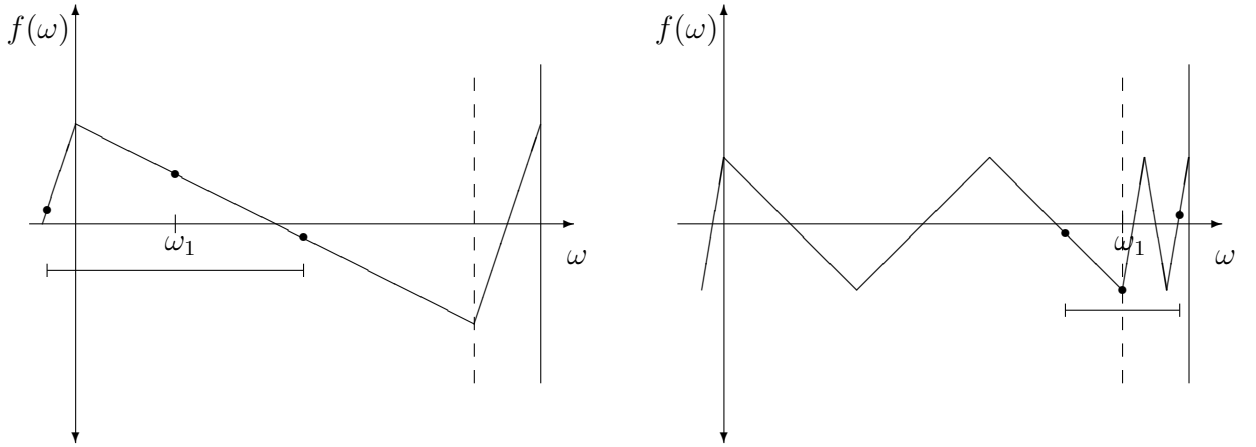


Figure 6: Categorization examples

version of our model with more than two periods both the number of categories and the nature of the categorization arise endogenously.

## 8 Discussion and Conclusion

The model we have analyzed sheds light on a number of phenomena. The first is path dependence, whereby there may be differences in performance based purely on the history of events. In the model, a series of “lucky” early realizations of the state can lead to a very effective rule book being established, delivering highly efficient outcomes. Conversely, bad draws early on can have persistent effects which cannot be overcome (recall figure 4). There is a sizeable empirical literature which documents what Gibbons (2006) calls “persistent performance differences among seemingly similar organizations”: that is, substantial differences in the performance of organizations which are difficult to account for (Mairesse and Griliches (1990), McGahan (1999), Chew et al. (1990)).

One potential theoretical explanation for this is that differences in performance correspond to different equilibria of a repeated game (Leibenstein (1987)). As Gibbons (2006) points out, such an explanation provides a basis for the existence of different equilibria, but is silent about the dynamic path of differences, and how one might switch equilibria. In contrast, our model exhibits path dependence. It can be the case that a firm has more efficient communication than another purely because of chance: they got some good draws

early on which allowed them to develop an effective rule book.

Furthermore, the model with multiple activities highlights that there can be a complicated evolution of governance structures because of communication needs. Various activities may or may not be delegated, and this can change over time as more effective rules are developed. At a minimum, this suggests that bounded communication is a potential explanation for the heterogeneity of observed organizational forms.

The model also provides a potential explanation for *learning curves*, whereby a firm's cost declines with cumulative output/experience. The fact that learning can take place at different rates depending on history or different abilities to overwrite is consistent with the evidence on heterogeneity of rates of learning (Argote and Epple (1990)).

Another empirical regularity which has been documented is that firms often “start-small” in the sense that they grow less quickly than they otherwise would. *Prima facie* this is puzzling, as profits are being foregone in the process. Watson (2002) argues that a rationale for starting small is to build cooperation in a low stakes environment and then enjoy the benefits of the cooperative equilibrium in a high stakes environment. Our model offers a different (and potentially complementary) notion of starting small. There is option value in developing rules and consequently rules can be under-inclusive, particularly early on (recall Proposition 2). One way to interpret the actions in our model that would make it connect with the starting-small idea would be to regard the zero expected payoff that obtains in states for which no rule has been defined as coming from a blanket instruction that the firm should decline any business opportunities that might arise which do not fall into any of the categories for which rules have been defined. With this interpretation, our model would be one in which the size of the business grows over time as new situations arise and enable the firm to promulgate rules about how it should exploit such opportunities.

The possibility that communication may start out vague and become more precise over time has the flavor of what Gibbons (2006) refers to as “building a routine”. The model also highlights that the nature and extent of inefficiency from bounded communication—and the degree to which rules are over or under inclusive—can depend importantly on the ability of the principal to amend previously developed rules. Indeed we saw that in the vast overwriting regime there was no option value from leaving rules undefined as they could

always be amended.

There are two obvious directions for further work. The first is to provide a more concrete foundation for the inability of the principal to describe the state to the agent. We remarked early on that we had in mind state spaces which were complex in some sense, though we conducted our analysis on the unit interval. It would be useful to understand what kinds of environments, or what other cognitive limitations, give rise to the inability to perfectly communicate.

Finally, the second direction would be to more explicitly model the tradeoff between the price mechanism and the authority relationship. This would allow one to address the question of when the price mechanism is superior at adapting to changing circumstances than the authority relationship: Barnard versus Hayek. What we have done is to highlight one particular cost of the authority relationship, and analyzed how different governance structures perform in that context. We remarked earlier that a complete understanding of the tradeoff between the efficiency of markets and firms in changing circumstances would seem to require a general model of price formation. That is, a model in which one can analyze how frictions affect the incorporation of information into the price mechanism and hence the circumstances in which the price mechanism works well and those where it does not. Although the literature on rational expectations equilibrium and securities markets provided some steps in this direction<sup>8</sup>, we are not aware of a suitable model for studying this question. For that model would not only need to address the question of how information is incorporated into prices, but also be able to rank economies according to the allocative efficiency effects of imperfect prices. Even the first question is daunting enough. Any model of price formation with rational expectations is subject to the Milgrom-Stokey No-Trade Theorem (Milgrom and Stokey (1982)), raising the awkward question of how information comes to be incorporated into prices if those who collect the information cannot benefit from it. Nonetheless, a complete model of price formation seems essential to satisfactorily answer the larger question of when markets or firms adapt better to changing circumstances.

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<sup>8</sup>For example: Grossman (1976), Allen (1981), Glosten and Milgrom (1985), Kyle (1985)

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