COMPETING AUCTIONS

Glenn Ellison
Massachusetts Institute of Technology

Drew Fudenberg
Harvard University

Markus Möbius
Harvard University

Abstract

This paper shows that larger auctions are more efficient than smaller ones, but that despite this scale effect, two competing and otherwise identical markets or auction sites of different sizes can co-exist in equilibrium. We find that the range of equilibrium market sizes depends on the aggregate buyer-seller ratio, and also whether the markets are especially "thin." (JEL: D44, L11)

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E-mail addresses: Ellison: gellison@mit.edu; Fudenberg: dfudenberg@harvard.edu; Möbius: mobius@fas.harvard.edu

1. Introduction

This paper examines a simple model of competing auction sites to give some insights into the concentration of auction markets. In our model, there are B ex-ante identical buyers, each with unit demand, and S sellers, each with a single unit of the good to sell and a reservation value of zero. At the start of the model, buyers and sellers simultaneously choose between two possible locations. Buyers then learn their private values for the good, and a uniform-price auction is held at each location. This is a very stark model, but we believe that it provides some useful insights, and that it serves as a benchmark case for richer and more realistic models.

In the last few years there have been a number of high-profile battles between competing auction sites. In 1998 and 1999, Yahoo! and Amazon, among others, launched online auction sites to compete with eBay. In 1999, literally hundreds of firms set up B-to-B auction sites. In the offline world, Bonham's recently engineered the merger of what may have been the third, fourth, and fifth largest fine arts auction houses in the world in an attempt to build a viable challenger to Sotheby's and Christie's.

These battles have often ended with a single overwhelming winner. For example, Yahoo!'s U.S. auction revenues are estimated to be at most 5% of eBay's and Amazon's are even lower.[1] In Japan, things tipped the other way: eBay entered 5 months after Yahoo!, was never able to garner a significant market share, and abandoned the market in February of 2002. The traditional fine arts auction market exhibits a less extreme and perhaps more intriguing form of concentration: Christie's and Sotheby's have jointly dominated the market for a

[1] Yahoo! abandoned its European auction operations in May of 2002.

century or more.[2] The market has not tipped against either of the leaders, each of which has come back from periods when it was somewhat behind, but no third firm has been able to challenge the leaders.

Why might auction activity concentrate? Some previous models have been built around an assumption of a preference for variety.[3] Our analysis starts from a simpler premise: A seller will choose the auction site where her expected price is highest, and a buyer will choose the auction site where his expected consumer surplus is highest. These preferences are largely opposed (high prices vs. low prices) so that the concentration of auction activity cannot be explained by a simple reference to network externalities. However, we show that sellers' and buyers' preferences are not completely opposed, because larger markets typically provide greater expected surplus per participant. We refer to this as the "scale effect" or "efficiency effect;" it makes concentration at a single auction site efficient, and so pushes the market toward concentration.[4]

Despite the scale effect, our model does not always tip to complete concentration, because of an offsetting force we call the "market impact effect." When a seller contemplates a switch from market 1 to market 2 she takes into account that her joining market 2 will increase the seller-buyer ratio there, and thereby lower the expected price.[5] The market impact effect is present in any model with a finite number of agents, and favors equilibrium multiplicity: If the

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[2] The U.S. Department of Justice estimated that in 1999 the combined market share of Christie's and Sotheby's was 90%. The two firms achieved prominent positions by the early 19th century. See Learmount (1985).
[3] These issues have been examined in a number of previous papers. For example, in Gehrig (1998), Stahl (1982), and Wolinsky (1983), consumers prefer larger markets because they have a finer grid of available "varieties."
[4] The scale effect implies that it is inefficient for there to be more than one active market. Note that trade in a given market is always ex-post efficient; in contrast to the literature on the asymptotic efficiency of double auctions and other exchange mechanisms with a fixed set of traders, e.g. Gresik and Satterthwaite (1989), Satterthwaite and Williams (1989), and Tatur (2001). These papers study trade in settings where no admissible mechanism is ex-post efficient, and derive bounds on the rate at which the ex-post inefficiency disappears as the economy grows.

decrease in price in market 2 that results from a seller switching from market 1 to market 2 is large enough to offset market 2's initial price advantage, then sellers in market 1 may happy to stay in market 1 while those in market 2 are happy to stay in market 2. Thus, whether the smaller market can survive depends on the relative magnitude of these two effects. As we will see, both effects become small as the markets grow, and since they both shrink at the same rate, neither one asymptotically dominates the other.

Our model can always account for a market being dominated by a single auction site. Our most striking conclusion, however, is that the model also has equilibria in which sites of quite different sizes coexist, and that the critical mass of buyers an auction site must attract to be viable increases proportionally with the total buyer population. To get the main ideas across more simply, many of our results will be about "quasi-equilibria," which differ from full equilibria only by not requiring that the numbers of agents in each market be integers. Section 3 defines the concept formally, and explains how the constraints that no agent wants to switch markets are related to whether the market impact effect is strong enough to outweigh the scale effect.

Section 4 analyzes the special case where buyers' valuations are uniformly distributed. We give explicit expressions for the strength of the market impact and scale effects, and find the quasi-equilibrium set. For any fraction between one-fourth and three-fourths, there is always a quasi-equilibrium with that fraction of the buyers at auction site 1. There can also be coexisting sites that are somewhat more different in size; just how unequal the two markets can be depends on the aggregate seller-buyer ratio, for reasons that we explain. We provide graphs illustrating how the set is determined by four curves. We also prove a result about full

[5] Bulow and Klemperer (1996) emphasize a related effect, the amount that a seller gains from attracting an extra buyer; they show that it is larger than the gain from implementing an optimal reserve price.

equilibria with coexisting sites. The result that competing auction sites with a size ratio of 3:1 or 4:1 may both be viable is consistent both with the dominance of eBay (and of Yahoo! in Japan) and with the long-term coexistence of Christie's and Sotheby's: In the on-line auction markets, the new entrants seem to have never reached the necessary size, while in the art market the two firms' relative market shares appear to have remained within the bounds that our result prescribes, and no third firm has ever achieved the requisite critical mass.

Section 5 contains a more general investigation of ex-ante payoffs and scale effects in single auctions. We provide some general results about large markets providing higher payoffs than small ones, derive asymptotic approximations to the buyers' and sellers' payoffs when the number of agents is large, and show that the ex-ante "inefficiency" of an auction with N participants (that is, its per-capita payoff compared to a larger auction with the same seller-buyer ratio) and the market impact effect both decline at rate $1/N$.[6]

Section 6 contains two results on competing auctions. The fact that inefficiency is of order $1/N$ and the market impact effect is also of order $1/N$ allows us to appeal to a general result in Ellison and Fudenberg (2003) and conclude that (when the number of buyers and sellers is large) the model has a large number of quasi-equilibria with two active auction sites. An interesting aspect of the quasi-equilibrium set is that the critical mass of buyers that an auction site must attract to be viable increases proportionately with increases in the total buyer population.

[6] Note that this makes the scale effect we study substantially larger than the ex post inefficiency discussed in Gresik and Satterthwaite (1989), Satterthwaite and Williams (1989), and Tatur (2001) in large economies. Our calculation uses similar techniques.

Section 7 extends our model to allow the competing auction sites to set prices.[7] As in the standard models of competing firms selling goods which provide network externalities, a critical factor is how buyers and sellers coordinate after the auction sites post prices. Without added assumptions, the model predicts that fees charged by the auction site can lie in a broad range, and that the possibilities for coexistence are as in our base model. If one adds the assumption that buyers and sellers always coordinate on an equilibrium that is Pareto optimal for them, the range of possible fees is much smaller, but the scope for multiple sites of different sizes coexisting is not reduced. Our analysis differs from past work on competition in auction rules in a number of ways. McAfee (1993), Peters and Severinov (1997), and Burguet and Sakovics (1999) study models in which buyers choose among auction sites, each of which has a single seller. These papers only examine equilibria in which buyers mix symmetrically, which rules out equilibria supported by market impact effects.[8] Caillaud and Jullien (2001, 2003) study a model of sites that try to attract participants on both sides of an interaction, and show that allowing negative prices can reduce the set of equilibria. They assume a particular specification of the payoffs that incorporates the idea that larger markers offer more "varieties" and hence better matches, and they have a continuum of participants on both sides of the interaction, which eliminates the market impact effect.

Section 8 examines another factor that might be thought to support concentration, namely market "thinness." One interesting aspect of eBay listings is that most of them seem to be unique items; this may be an important common trait of fine art and online auctions.

[7] We consider only simple pricing schemes and not more complicated schemes as in Caillaud and Jullien (2003) where buyers and sellers may be offered contracts that only entail fees if a specified number of other buyers and sellers attend.

2. The Model

In our model, there are B buyers, S sellers, and S units of a single good. The sellers are risk neutral, have zero reservation value, and are endowed with one unit of the good.[9] Buyers receive utility from consuming one unit of the good. The buyers' values v are identically and independently distributed with a cumulative distribution function F, which has a three-times continuously differentiable density f that is positive on its support $[0,\overline{b}]$, with $\overline{b}$ allowed to be infinite.[10] We assume that the expected value Ev is finite.

We model location choice with a two stage game. In the first stage, which occurs before the buyers learn their valuations for the good, buyers and sellers simultaneously choose whether to attend market 1 or market 2. One situation in which buyers would choose locations before learning their valuations is when buyers need to go to the auction site and inspect the good to learn their valuation. Another situation where this assumption is appropriate is when the buyers are dealers who participate in many auctions, have time-varying idiosyncratic valuations due to inventory or other factors, and who choose a single auction site for all of their purchases because they can save on transactions costs or build a better reputation. In the second stage, buyers learn their valuations and a uniform price auction occurs in each market. Thus, if a market with S sellers and B buyers does not have excess supply, the price there is the $S+1$st highest of the B buyer values. We use p or $v^{S+1:B}$ to denote this price. More generally, we denote the kth highest order statistic of a draw of n values as $v^{k:n}$.[11] Our assumption that v has a

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[8] Peters and Severinov do discuss some potential implications of buyers' using asymmetric pure strategies in their conclusion.

[9] Section 5 briefly discusses the extension to the case of a common positive reservation value.

[10] The propositions of this section extend to distributions without densities, although some strict inequalities become weak, and the notation and proofs become a bit more complicated due to the need to account for ties. The assumption that f has a third derivative is used only in large-economy approximations.

[11] This is a slight twist on the usual notation in textbooks, where "first" order statistic is the smallest.

finite expectation implies that the expectations of the order statistics are finite too. We write $f^{k:n}$ for the density of $v^{k:n}$.

We assume that $S+1<B$ so that if all agents go to a single market there is excess demand and so with probability one the market price is strictly positive. A seller's expected utility is just the expected price in the market he or she chooses, $u_s(S,B)=\mathrm{E}(v^{S+1:B})\equiv \overline{p}(S,B)$. The expected utility of a buyer who attends a market with *S* sellers and *B* buyers (including himself) is $u_b(S,B)=\mathrm{E}(v-v^{S+1:B}\mid v\geq v^{S:B})\Pr(v\geq v^{S:B})$. The surplus per seller (or per item sold) is $\mathrm{E}(v\mid v\geq v^{S:B})=\mathrm{E}(v\mid v>v^{S+1:B})$, which we define to be $w(S,B)$; note that

$$w(S,B)=\int_0^{\overline{b}}\left(\int_x^{\overline{b}} vf(v\mid v>x)dv\right)f^{S+1:B}(x)dx.$$

3. Equilibrium Conditions: The Market Impact and Efficiency Effects

In this section we present a very simple result to illustrate that in our model whether two active auction sites can coexist (and how different in size they may be) depends on the relative strength of two factors: the "efficiency" advantage of a larger site and the adverse "market impact" that a buyer or seller has when he or she switches sites.

Consider the possibility of a pure strategy equilibrium with S_1 sellers and B_1 buyers in market 1 and S_2 sellers and B_2 buyers in market 2, where $S_1+S_2=S$ and $B_1+B_2=B$. This will be an equilibrium if and only if five constraints are satisfied:

(S1) $u_s(S_1,B_1)\geq u_s(S_2+1,B_2)$

(S2) $u_s(S_2,B_2)\geq u_s(S_1+1,B_1)$

(B1) $u_b(S_1,B_1)\geq u_b(S_2,B_2+1)$

(B2) $u_b(S_2, B_2) \geq u_b(S_1, B_1+1)$

(I) S_1, S_2, B_1, and B_2 are nonnegative integers.

We will often find it convenient to ignore the integer part of the final constraint and give results characterizing what we call "quasi-equilibria".

DEFINITION 1. A *quasi-equilibrium* is a vector of nonnegative real numbers (S_1, S_2, B_1, B_2) with $S_1 + S_2 = S$ and $B_1 + B_2 = B$ that satisfy (S1), (B1), (S2), and (B2).

Our first result is an algebraically trivial restatement of the quasi-equilibrium conditions that provides a useful perspective on when they will and will not hold.

PROPOSITION 1. *A vector of nonnegative real numbers* (S_1, S_2, B_1, B_2) *with* $S_1 + S_2 = S$ *and* $B_1 + B_2 = B$ *is a quasi-equilibrium if and only if it satisfies the following four constraints:*

(B1′) $u_b(S_2, B_2) - u_b(S_2, B_2+1) \geq u_b(S_2, B_2) - u_b(S_1, B_1)$

(S1′) $u_s(S_2, B_2) - u_s(S_2+1, B_2) \geq u_s(S_2, B_2) - u_s(S_1, B_1)$.

(B2′) $u_b(S_1, B_1) - u_b(S_1, B_1+1) \geq u_b(S_1, B_1) - u_b(S_2, B_2)$

(S2′) $u_s(S_1, B_1) - u_s(S_1+1, B_1) \geq u_s(S_1, B_1) - u_s(S_2, B_2)$.

The left-hand sides of the two stay-in-market-1 conditions, (B1′) and (S1′), measure the "market impact" that agents have when they move to market 2. The right-hand sides measure the degree to which market 2 is more attractive to buyers and sellers, respectively, given the current division of buyers and sellers. Note that when $S_1 = S_2$ and $B_1 = B_2$ the right-hand sides of the four constraints are all zero, while the left-hand sides are all strictly positive; this reflects the fact that an equal split between the two sites is a strict equilibrium.

As we discuss in detail in Section 5, larger markets typically yield higher per-capita payoffs, because they come closer to the ideal of allocating the good to a buyer if and only if his valuation is high. This scale effect implies that one or both of the right-hand sides of (B1') and (S1') will be positive when market 2 is larger than market 1. Proposition 1 says that equilibrium requires that the market impact effects offset the higher payoffs that a larger market provides.

As the markets become more different in size, the advantage of the larger market grows. How different in size the two sites can be in equilibrium will depend on how strong the scale effect is relative to the market impact effect. If the scale effect is strong and payoffs increase rapidly with increases in market size, then the smaller site will need to be nearly as large as the larger one to be viable. If they do not (or the market impact effects are extremely strong), then auction sites of very different sizes may be able to coexist.

4. The Model with Uniformly-Distributed Valuations

In this section we explicitly calculate the efficiency and market impact effects and derive the implications for the coexistence of auction sites of different sizes under the assumption that buyer valuations are uniformly distributed. Qualitatively, the main conclusions are that markets of quite different sizes can coexist, that an auction site does need a "critical mass" of participants to be viable, and that the critical mass is roughly a specified *fraction* of the number of participants at the larger auction site (independent of the total number of participants). The example is intended to provide a straightforward illustration of more general results we'll derive in the next section. The assumption of uniformly distributed valuations may also be a reasonable one to use to think about auctions for goods like beanie babies, where the option of purchasing from a retail store puts an upper bound on the amount that any buyer would bid at auction.

4.1 Quasi-equilibria

The utility functions for uniformly distributed valuations are simple to derive:

PROPOSITION 2. *When buyer valuations are uniform on [0,1], the utility functions are* $u_s(S,B) = \frac{B-S}{B+1}$ *and* $u_b(S,B) = \frac{S(1+S)}{2B(B+1)}$.

Proof. The *i*th lowest of *n* draws from a uniform distribution is distributed Beta(*i, n-i+1*) and has expectation $i/(n+1)$ (see e.g. David (1970).) The seller's expected utility is equal to the expectation of the price. The price is the $S+1^{\text{th}}$ highest buyer value, which is the $B-S^{\text{th}}$ lowest. Hence, $u_s(S,B) = \overline{p} = \frac{B-S}{B+1}$. Because a buyer's valuation conditional on being greater than p is uniform on $(p,1]$, a buyer's expected utility conditional on winning the good at p is $(1-p)/2$. Each buyer wins the good with probability S/B, so the buyer's expected utility is $u_b(S,B) = S\left(1-\overline{p}(S,B)\right)/2B = S(1+S)/2B(1+B)$. QED

Using these expressions it is easy to calculate compare the payoffs of two markets with the same seller-buyer ratio. Note that although a larger market yields higher payoffs per capita, buyers are actually better off in the smaller market. Note also that the market impact of adding another buyer is strongest when S/B is near to 1.

PROPOSITION 3. *Suppose buyer valuations are uniform on [0,1].*

(a) The per-seller scale advantage of the larger market is

$$w(\gamma B_2, B_2) - w(\gamma B_1, B_1) = \frac{(1-\gamma)}{2}\frac{(B_2 - B_1)}{(B_2+1)(B_1+1)}.$$

The payoff advantage/disadvantage of the larger market for the sellers/buyers is

$$u_s(\gamma B_2, B_2) - u_s(\gamma B_1, B_1) = (1-\gamma)\frac{(B_2 - B_1)}{(B_2+1)(B_1+1)},$$

$$u_b(\gamma B_2, B_2) - u_b(\gamma B_1, B_1) = -\frac{\gamma(1-\gamma)}{2}\frac{(B_2 - B_1)}{(B_2+1)(B_1+1)}.$$

(b) The market impact effects for both $j=1$ *and* $j=2$ *are given by*

$$u_s(S_j, B_j) - u_s(S_j+1, B_j) = \frac{1}{B_j+1},$$

$$u_b(S_j, B_j) - u_b(S_j, B_j+1) = \frac{S_j(S_j+1)}{B_j(B_j+1)(B_j+2)}.$$

Proof. The results follow immediately from the formulas in Proposition 2. QED

To determine how different in size two viable auction sites can be when the total number of buyers and sellers are *B* and *S*, respectively, it is sufficient to find the smallest and largest values of B_1 for which the market impact effect is strong enough to outweigh the payoff advantage of the larger market. Note that when trying to construct an equilibrium, it may be helpful to place relatively more buyers in the larger market , so that the seller's market impact need only outweigh a fraction of the larger market's efficiency advantage, rather than something that is larger than the efficiency advantage.

The exact formula for the critical mass of buyers necessary to make an auction site viable is too complicated to be enlightening. Proposition 4 gives an accurate and intuitive approximation. Roughly, the proposition says that the fraction of the buyers that the smaller auction site must attract to be viable is about $(1/4)(1-S/B)$. When S/B is small this means that an auction site must attract about one-quarter of the buyers to survive. When S/B is close to one it implies that a site with a tiny fraction of the participants can survive alongside a much larger site.

PROPOSITION 4: *Fix B and S with* $B > S + 2$. *When buyer values have the uniform distribution, there is an unique* $\underline{B}_1 \in [0, B/2]$ *for which there is an* S_1 *such that (S1) and (B1) both hold with equality at* $(S_1, S-S_1, \underline{B}_1, B-\underline{B}_1)$. *There exists an* S_1 *such that* $(S_1, S-S_1, B_1, B-B_1)$ *is a quasi-equilibrium if and only if* $B_1 \in [\underline{B}_1, B-\underline{B}_1]$. *Moreover,*

$$-\frac{5}{4B} < \frac{\underline{B}_1}{B} - \frac{1}{4}\left(1-\frac{S}{B}\right) < \frac{3}{4B} + \frac{1}{S}.$$

Sketch of proof: Intuitively, one might expect that the binding constraints determining whether $(S_1, S-S_1, B_1, B-B_1)$ is a quasi-equilibrium are always (S1) and (B1) when $B_1 < B/2$, i.e. that the binding constraint is to keep participants from leaving the smaller, less efficient market. It turns out that this is not quite right, but it is true that the set of values of B_1 with $B_1 < B/2$ for which there is some S_1 that gives a quasi-equilibrium is indeed the set of B_1 for which (S1) and (B1) can be satisfied simultaneously. The first step in our proof (presented in the appendix) is a lemma showing that this is true in any model satisfying three conditions: a monotonicity condition requiring that sellers (buyers) like sites with more buyers (sellers) and fewer sellers (buyers); a "large-market-efficiency" condition which requires that either sellers or buyers (or both) prefer site 2 to site 1 if site 2 has more buyers; and a set of boundary conditions.

We show that the model with uniformly distributed buyer values satisfies these three conditions. All that is required to complete the proof is then to analyze the conditions that are required for the (S1) and (B1) constraints to hold simultaneously. We show that $\underline{B}_1$ is the solution to a particular quadratic equation and derive the bounds given in the proposition by approximating the solution. QED

Here is a rough sketch that provides some intuition for the role of the aggregate seller/buyer ratio $S/B = \gamma$ in the limiting value of $\underline{B}_1$. When both markets are large, the market impact effects are small, so the seller/buyer ratios in each market must be about the same. From Proposition 3 we know that on a per-buyer basis the scale advantage is about $.5\gamma(1-\gamma)(1/B_1 - 1/B_2)$. Since (S1) and (B1) hold with equality, both buyers and sellers are better off in the larger market. To have a quasi-equilibrium, the scale advantage must be offset by the market impacts that buyers and sellers have when moving to the larger market. Proposition 3 shows that the seller's market impact is about $1/B_2$, and the buyer's is about γ^2/B_2. Hence, the "best" way (for the purpose of constructing an equilibrium) to satisfy both (S1) and (B1) is to choose the seller-to-buyer ratio so that the large market's payoff advantage for a buyer is about γ^2 times as large as the large market's payoff advantage for a seller. A few calculations show that this is consistent with the size of the scale advantage if

$(1+\gamma)/(1-\gamma) \geq 2(B_2 - B_1)/B_1$; this is equivalent to our conclusion that for large B, B_2 can be about $(3+\gamma)/4$.

4.2 Illustrative figures

Figures 1 and 2 illustrate the structure of the equilibrium and quasi-equilibrium sets for the uniform distribution with ten buyers and five sellers. Figure 1 graphs the fractions of sellers in market 1 that make buyers and sellers exactly indifferent between the two markets against the fraction of buyers in market 1. The solid curve, which is the locus where buyers' utility is equal in both markets, lies above the dotted curve, which is the locus where sellers' utility is equalized in both markets, when $B_1 < B/2$. The unique intersection of the curves is at $B_1 = B/2$. If buyers and sellers did not adversely affect prices when moving to the other market, the only quasi-equilibrium with split markets would be an unstable equilibrium with an exactly 50-50 split between the two markets.

Figure 2 graphs the values of S_1/S for which the (B1), (B2), (S1) and (S2) constraints hold with equality for the same utility functions as in Figure 1. The quasi-equilibrium set is the parallelogram-shaped region in the center of the figure below the curves where (S1) and (B2) hold with equality and above the curves where (S2) and (B1) hold with equality. In this example, quasi-equilibria exist whenever the smaller market has at least 11% of the buyers (meaning B_1 is at least 1.1). We have placed small stars in the figure at points within the quasi-equilibrium set where the numbers of buyers and sellers are both integers. These are the equilibria. In an equilibrium the smaller market can have two buyers and one seller or four buyers and two sellers. There is no equilibrium with three or five buyers in the smaller market.

With three buyers in the smaller market, for example, then there are a range of values of S_1 near one-and-a-half which satisfy the quasi-equilibrium conditions. None of these allocations, however, satisfy the integer constraints – sellers would be unwilling to stay in small market if there were two sellers, while buyers would be unwilling to stay in a market if there was only one seller.

To illustrate how markedly the scale effect declines with the size of the market, Figure 3 graphs the equal-buyer-utility and equal-seller-utility curves for a model with 30 buyers and 15 sellers and a uniform distribution of seller valuations (as in Figure 1). The curves are much closer together than the curves in Figure 1. Sellers are indifferent when prices are equal in the two markets. The closeness of the two curves reflects that efficiency differences are fairly small and can only offset a small difference in price. Figure 4 graphs the four curves that bound the quasi-equilibrium set in this case. One interesting thing to note is that the range of market sizes in the quasi-equilibrium set is very similar to that in Figure 2: Here a quasi-equilibrium exists whenever at least 12% of the buyers are in the small market, as compared to the 11% in Figure 2. The quasi-equilibrium set looks much flatter in the *S*-dimension. This reflects that the market impact is much smaller and hence buyer and seller utility (the latter of which is equal to the price) have to be more nearly equal in the two markets in equilibrium. The stars in the figure illustrate that there are nonetheless a substantial number of true equilibria.

4.3 True equilibria

So far we have been ignoring the constraint that the numbers of buyers and sellers in each marker should be an integer. With a small number of traders it may be that only a few ratios of markets sizes are possible. However, one would expect these integer problems to become less important in large markets. Our next result is a demonstration that for any "target" value for the seller-buyer ratio and any target value for B_1 / B, one can for any sufficiently large B always find an S with S / B close to the target for which the model with S sellers and B buyers has a true equilibrium with the fraction of buyers at site 1 being approximately equal to the target value for B_1 / B. The statement of the result uses α as the target level of $\frac{B_1}{B}$ and γ as the target level of S / B.

PROPOSITION 5: *For any target market ratios* α, γ *with* $0 < \gamma < 1$ *and* $\alpha \in \left(\frac{1}{2} - \frac{\gamma}{2}, \frac{1}{2} + \frac{\gamma}{2}\right)$, *and any* $\varepsilon > 0$, *there exists* $\underline{B}$ *such that for all* $B > \underline{B}$ *there is an equilibrium* (S_1, S_2, B_1, B_2) *with* $B_1 + B_2 = B$, $\left|B_1 / B - \alpha\right| < \varepsilon$, *and* $\left|S / B - \gamma\right| < \varepsilon$.

Proof: See appendix.

The proof first constructs a quasi-equilibrium with equal prices that approximates the target ratios, but where only B_1 and B_2 are guaranteed to be integers; we then use this partition to construct an integer-valued partition where all of the incentive constraints are satisfied but prices are only approximately equal.

5. Ex-Ante Payoffs and Scale Effects in a Single Auction

This section drops the restriction to uniformly distributed buyer valuations and analyzes how ex-ante payoffs vary with market size in a single market. Understanding this scale effect is important for determining when a larger market will drive out a smaller one, and we think it is also interesting in its own right.

Consider a market with S sellers and $B > S$ buyers. Ex-post efficiency requires that the buyers with the S highest values receive the good, so the expectation of the maximum possible total surplus is $B\,\Pr(v \geq v^{S:B})\mathrm{E}(v \mid v \geq v^{S:B}) = S\mathrm{E}(v \mid v \geq v^{S:B})$; this surplus is realized by the uniform-price auction that we defined above. Note that as the market grows, holding the buyer-seller ratio fixed, the auction's outcome converges to a deterministic limit, with the good given to all buyers whose value exceeds the market-clearing price, which is $\overline{v} \equiv F^{-1}\left(1 - S/B\right)$; another way of saying this is that the auction converges to the competitive equilibrium for the limiting "continuum economy." Welfare per seller converges to the average of the buyer values on the range where value is at least the market price, $\mathrm{E}\left[v \mid v \geq \overline{v}(S/B)\right]$; denote this $w_\infty(S/B)$.

PROPOSITION 6: *With an efficient allocation of goods to buyers, the expected surplus per seller in a finite market with S sellers and B buyers is strictly less than the expected surplus per seller in a market with continua of buyers and sellers and the same seller-buyer ratio.*

The idea of the proof is simply that the distribution of realized utility as a function of buyer's value in the large market first-order stochastically dominates the distribution in the small one. In both markets, buyers have the same probability of receiving the good, but in the small market, buyers do so in a less efficient way, as they sometimes receive the good when $v < \overline{v}$.

Proof: Since we are holding the seller-buyer ratio fixed, it will be sufficient to prove that surplus per buyer is higher in the large market. Let $c(v)$ be the buyer's probability of consuming the good in the finite market when his value is *v*. Then the expected surplus per buyer in the continuum market is $\mathrm{E}(v \mid v \geq \overline{v})\Pr(v \geq \overline{v})$, and the expected surplus per buyer in the finite market is $E(v \bullet c(v))$. The difference in per buyer surplus between the large and small market is

$$\begin{aligned} &\mathrm{E}\left(v(1-c(v)) \mid v \geq \overline{v}\right)\Pr(v \geq \overline{v}) - \mathrm{E}(vc(v) \mid v \leq \overline{v})\Pr(v \leq \overline{v}) \\ &> \mathrm{E}\left(\overline{v}(1-c(v)) \mid v \geq \overline{v}\right)\Pr(v \geq \overline{v}) - \mathrm{E}(\overline{v}c(v) \mid v \leq \overline{v})\Pr(v \leq \overline{v}) \\ &= \overline{v}\left[\Pr(v \geq \overline{v}) - \mathrm{E}(c(v))\right] = \overline{v}\left[S/B - S/B\right] = 0. \end{aligned}$$

The strict inequality follows from the fact that $c(v)$ is strictly between 0 and 1 on for $v \in (0, \overline{v})$, which in turn follows from the assumptions that $B > S$ and the cumulative distribution of buyer values is strictly monotone. QED.

The next result extends the previous one by showing that the expected surplus per agent is monotone in market size.

PROPOSITION 7: *If m and n are integers with m<n, then* $w(mS, mB) < w(nS, nB)$.

Proof: See appendix.[12]

[12] We thank Jonathan Weinstein for the proof of this proposition.

We should emphasize that these results are about the expected per-capita surplus, and do not imply that both buyers and sellers must become better off as the market grows, as the size of the market can influence the allocation of the surplus as well as its level. Indeed, we will see that there are cases where buyers strictly prefer smaller markets, holding the seller/buyer ratio fixed.

We now present results providing approximations to the payoffs obtained by buyers and sellers in large auctions. The results are asymptotic approximations relevant in the limit as the number of buyers and sellers grows with the seller-to-buyer ratio held fixed at γ. This result may be of interest on its own; it will also be useful in our analysis of two competing large markets.

PROPOSITION 8: *The utility functions of the auction model have extensions to the domain* $\mathbb{R}\times\mathbb{R}$ *that are continuous and such that for all* $\gamma \in (0,1)$ *the approximations*

$$u_s(\gamma B, B) = F^{-1}(1-\gamma) - \left((1-\gamma)\frac{2f(\overline{v})^2 + \gamma f'(\overline{v})}{2f(\overline{v})^3} \right) / B + o(1/B)$$

$$u_b(\gamma B, B) = \gamma E(v - \overline{v} \mid v \geq \overline{v}) + \left(\gamma(1-\gamma)\frac{f(\overline{v})^2 + \gamma f'(\overline{v})}{2f(\overline{v})^3} \right) / B + o(1/B)$$

hold uniformly in γ *when* B *is large.* [13]

Proof: See appendix.

To see why something like this result might be true, consider the formula for the seller's utility $u_s(\gamma B, B)$:

$$u_s(\gamma B,B) = E\left[v^{S+1:B}\right] = \int_0^{\bar{b}} v\left[\binom{B}{\gamma B}(B-\gamma B)F(v)^{B-\gamma B-1}\left(1-F(v)\right)^{\gamma B} f(v)\right]dv$$

As $B \to \infty$, the density term in brackets becomes increasingly concentrated around its maximum, which occurs at $v = \bar{v} = F^{-1}(1-\gamma)$. Hence, $u_s(\gamma B, B)$ tends to $F^{-1}(1-\gamma)$. Moreover, the tails of the density are bounded by an exponentially decreasing function of B. This observation and a change of variables lets us show that the rate at which the integral approaches its limit is determined by the behavior of $F^{-1}(y)$ around $1-\gamma$, and this allows us to use a Taylor expansion of $F^{-1}(y)$ to determine the rate of convergence. Since the integral of the third-order term of the Taylor expansion is already $o\left(\frac{1}{B}\right)$, this rate depends only the first and second-order derivatives of F.

One immediate consequence of Proposition 8 is a result on ex-ante efficiency: the (per-seller) inefficiency of a market relative to the continuum limit shrinks at rate $1/B$.

COROLLARY 1: $w_\infty(\gamma) - w(\gamma B, B) = \dfrac{1-\gamma}{2f(\bar{v})}\dfrac{1}{B} + o(1/B).$

[13] More formally, when we say that an approximation $H(\gamma, B) = H_1(\gamma) + H_2(\gamma)/B + o(1/B)$ holds uniformly in γ when B is large, we mean that there exists a function $m_s(B)$ with $\lim_{B\to\infty} m_s(B) = 0$ such that

Proof: The result follows immediately from $w(\gamma B, B) = u_s(\gamma B, B) + (1/\gamma)u_b(\gamma B, B)$. QED

A second interesting consequence of Proposition 8 is that it illustrates that sellers and buyers have conflicting preferences about market size when the seller-buyer ratio is held fixed. The first term in the expressions for u_s and u_b are each agent's payoff in the continuum limit: the sellers' expected payoff approaches $F^{-1}(1-\gamma) = \overline{v}$ and the buyers' expected payoff approaches $\gamma E(v - \overline{v} \mid v \geq \overline{v})$. When $f'(\overline{v})$ is not too negative, the term of order $1/B$ in the expression for the seller's utility is negative, and the corresponding term in the expression for the buyer's utility is positive. This implies that when comparing two large markets sellers will prefer the larger market and buyers will prefer the smaller market.[14] Although the proposition only applies when the number of agents is sufficiently large, recall that under the uniform distribution (where $f'(v)$ is identically 0) buyers always preferred a smaller market. Schwartz and Ungo (2002) provide a more general result along these lines, allowing both for a broad class of distributions and making comparisons across auctions with nonidentical seller-to-buyer ratios.[15]

We believe that Proposition 8 extends to the case where all sellers have a common reservation value $\eta > 0$, provided that η is small enough that $\eta < \overline{v} = F^{-1}(1-\gamma)$. In this case, the reservation value is irrelevant in the continuum limit, and in finite markets the probability that a seller ends up keeping her unit goes to zero at an exponential rate. When $\eta > \overline{v}$, a positive fraction of sellers keep their units in the continuum limit, and the approximations in

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$| B(H(\gamma, B) - H_1(\gamma)) - H_2(B) | < m_s(B)$ for all $\gamma \in \Gamma$ and all integers B.

[14] Conversely, when $f'(\overline{v})$ is sufficiently negative, sellers prefer the smaller market.

Proposition 8 fail. In either case, the size of the quasi-equilibrium set is determined by the relative sizes of the market impact and scale effects, but it is only in the case $\eta < \overline{v}$ that we know that the range of market sizes given in Proposition 10 still applies.

6. Competing Auction Sites in Large Economies

In this section we present two results characterizing the quasi-equilibrium set under general assumptions about the distribution of buyer valuations. Our main result examines what happens when the number of buyers and sellers is large.

The market impact effect plays a critical role in letting multiple auction sites coexist. We first present a simple impossibility result to illustrate this: A market of fixed finite size could not survive if agents had the opportunity to attend a market with a continuum of buyers and sellers instead. Let the finite market be market 1, with numbers S_1 sellers and B_1 buyers, and let market 2 have a continuum of participants with seller-buyer ratio S_2 / B_2. The deterministic price $\overline{p}^2$ in market 2 is defined by $1 - F(\overline{p}^2) = S_2 / B_2$. A buyer or seller moving into the large market will have no effect on the price there, so for both markets to coexist it is necessary that the expected price in market 1 satisfies $\overline{p}^1 \geq \overline{p}^2$ (or else sellers move to market 2) and that $u_b^1 \geq u_b^2$ (or else the buyers move).

PROPOSITION 9: *There is no equilibrium in which trade takes place in both a continuum market and a finite one.*

[15] Schwartz and Ungo also discuss the implications this result may have for the mergers of auction houses if buyers have entry costs and a reduction in buyer utility could lead to lower buyer participation.

Proof: As before, let $c(v)$ be the buyer's probability of consuming the good in the small market when her type is *v*. Then $u_b^2 = \mathrm{E}(v - \overline{p}^2 \mid v \geq \overline{p}^2)\Pr(v \geq \overline{p}^2)$ is the buyer's expected utility in the continuum market, and

$u_b^1 = \mathrm{E}(vc(v)) - \frac{S}{B}E(p^1) = E(vc(v)) - E(E(p^1)c(v)) = E((v - \overline{p}^1)c(v))$ is the buyer's expected utility in the finite market. Sellers are willing to remain in the finite market only if $\overline{p}^1 \geq \overline{p}^2$. When this holds, however, we have

$u_b^1 = E((v - \overline{p}^1)c(v)) \leq E((v - \overline{p}^2)c(v)) < E((v - \overline{p}^2)I(v \geq \overline{p}^2)) = u_b^2$, so that buyers are not willing to remain in the finite market. QED

We now present the most important result of this section: a general result on the set of quasi-equilibria of large economies that complements our detailed analysis of the case of uniformly-distributed values. This result follows from Proposition 8 and a general result in Ellison and Fudenberg (2003). That paper uses the following assumption to show that when the economy is large, the equilibrium with two equal-sized markets is not an isolated "knife-edge," but that instead there is a "plateau" of quasi-equilibria:

CONDITION A4: There is a non-empty interval $\Gamma = [\underline{\gamma}, \overline{\gamma}] \subset (0, \infty)$ and twice continuously differentiable functions $u_s^*, u_b^*, G_s,$ and G_b on Γ with $du_s^* / d\gamma < 0$ and $du_b^* / d\gamma > 0$ such that the approximations

$$u_s(\gamma B, B) = u_s^*(\gamma) - G_s(\gamma)/B + o(1/B)$$

$$u_b(\gamma B, B) = u_b^*(\gamma) - G_b(\gamma)/B + o(1/B)$$

hold uniformly in γ when B is large.

PROPOSITION (Ellison and Fudenberg (2003)): *Assume Condition A4. Then, for any* $\varepsilon > 0$ *there exists a* $\underline{B}$ *such that for any integer* $B > \underline{B}$ *and any integer* S *with* $S / B \in \Gamma$, *the model with B buyers and S sellers has a quasi-equilibrium with* B_1 *buyers in market 1 for every* B_1 *with* $B_1 / B \in [\alpha^*(\gamma)+\varepsilon, 1-\alpha^*(\gamma)-\varepsilon]$, *where* $\gamma = S / B$ *and* $\alpha^*(\gamma) = \max\{0, \frac{1}{2} - \frac{1}{2r^*(\gamma)}\}$ *for*

$$r^*(\gamma) = \max\left(\left|\frac{2G_s(\gamma)}{-u_s^*(\gamma)}+1\right|, \left|\frac{2G_b(\gamma)}{\gamma u_b^*(\gamma)}+1\right|\right).$$

Proposition 8 shows that Condition A4 is satisfied in our auction model, which implies:

PROPOSITION 10: *For any* $\varepsilon > 0$ *there exists a* $\underline{B}$ *such that for any integer* $B > \underline{B}$ *and any integer* S *with* $S / B \in [\varepsilon, 1-\varepsilon]$, *the model with B buyers and S sellers has a quasi-equilibrium with* B_1 *buyers in market 1 for every* B_1 *with* $B_1 / B \in [\alpha^*(\gamma)+\varepsilon, 1-\alpha^*(\gamma)-\varepsilon]$, *where* $\alpha^*(\gamma) = \max\{0, \frac{1}{2} - \frac{1}{2r^*(\gamma)}\}$ *for*

$$r^*(\gamma) = \max\left(\left|\frac{(1-\gamma)\left(2f(\overline{v})^2 + \gamma f'(\overline{v})\right)}{f(\overline{v})^2}+1\right|, \left|\frac{(1-\gamma)\left(f(\overline{v})^2 + \gamma f'(\overline{v})\right)}{-\gamma f(\overline{v})^2}+1\right|\right).$$

Proof: Since $u_s^*(\gamma) = \bar{v}(\gamma) = F^{-1}(1-\gamma)$, $u_s^*{}'(\gamma) = -dF^{-1}(1-\gamma)/d\gamma = -1/f(\bar{v})$. And

$u_b^* = E(v-\bar{v} \mid v \geq \bar{v})\Pr(v \geq \bar{v}) = \int_{\bar{v}}^{\bar{b}} (v-\bar{v})f(v)dv$, so $u_b^*{}'(\gamma) = -(1-F(\bar{v}))d\bar{v}(\gamma)/d\gamma = \gamma/f(\bar{v})$.

Substitution of $G_b(\gamma) = -\gamma(1-\gamma)\dfrac{f(\bar{v})^2 + \gamma f'(\bar{v})}{2f(\bar{v})^3}$ and $G_s(\gamma) = (1-\gamma)\dfrac{2f(\bar{v})^2 + \gamma f'(\bar{v})}{2f(\bar{v})^3}$ in the general formula for *r* in the previous theorem yields

$$r^*(\gamma) = \max\left(\left|\frac{2G_s(\gamma)}{-u_s^{*'}(\gamma)} + 1\right|, \left|\frac{2G_b(\gamma)}{\gamma u_b^{*'}(\gamma)} + 1\right|\right)$$

$$= \max\left(\left|\frac{(1-\gamma)\dfrac{2f(\bar{v})^2 + \gamma f'(\bar{v})}{f(\bar{v})^3}}{1/f(\bar{v})} + 1\right|, \left|\frac{-\gamma(1-\gamma)\dfrac{f(\bar{v})^2 + \gamma f'(\bar{v})}{f(\bar{v})^3}}{\gamma^2/f(\bar{v})} + 1\right|\right)$$

$$= \max\left(\left|\frac{(1-\gamma)\left(2f(\bar{v})^2 + \gamma f'(\bar{v})\right)}{f(\bar{v})^2} + 1\right|, \left|\frac{(1-\gamma)\left(f(\bar{v})^2 + \gamma f'(\bar{v})\right)}{-\gamma f(\bar{v})^2} + 1\right|\right)$$

QED

To illustrate Proposition 10, consider the uniform distribution. Here, we have $r^*(\gamma) = \max\{|3-2\gamma|, |2-1/\gamma|\}$. When $\gamma > (5-\sqrt{17})/4 \cong 0.22$, the first term is larger and $\alpha^*(\gamma) = \dfrac{1}{2} - \dfrac{1}{2(3-2\gamma)} = \dfrac{1-\gamma}{3-2\gamma}$. When γ is close to 1, this is close to zero, so the conclusion of Proposition 10 in this case is close to the characterization of the quasi-equilibrium set given by Proposition 4. For small γ, $\alpha^*(\gamma) = \dfrac{1}{2} - \dfrac{1}{2(1/\gamma - 2)} = \dfrac{1-3\gamma}{2-4\gamma}$, which is close to ½ when γ is close to zero. Hence, Proposition 10 only establishes the existence of a tiny quasi-equilibrium plateau, whereas we know from Proposition 4 that the fraction of buyers that can be in market

1 in a quasi-equilibrium is about (1/4, 3/4). The difference is due to Proposition 10's only considering the possibility of quasi-equilibria with equal seller-buyer ratios is the two markets.

Next consider the exponential distribution, $f(v) = \exp(-v)$ for $v \geq 0$. Here $\bar{v}(\gamma) = -\ln\gamma$, and

$$r^*(\gamma) = \max\left(\left| \frac{(1-\gamma)\left(2e^{-2\bar{v}} - \gamma e^{-\bar{v}}\right)}{e^{-2\bar{v}}} + 1 \right|, \left| \frac{(1-\gamma)\left(e^{-2\bar{v}} - \gamma e^{-\bar{v}}\right)}{-\gamma e^{-2\bar{v}}} + 1 \right| \right) =$$

$$\max\left(2-\gamma, 1\right) = 2-\gamma \text{, so } \alpha^*(\gamma) = \frac{1}{2} - \frac{1}{2(2-\gamma)}.$$

As $\gamma \to 1$ the fraction of buyers that can be in market 1 in a quasi-equilibrium converges to $(0,1)$ as in the uniform case; as $\gamma \to 0$, the range of quasi-equilibria from Proposition 10 is $\left(1/4, 3/4\right)$. The 2002 version of this paper analyses case the exponential case in more detail; it shows that when the seller-buyer ratios are allowed to differ in the two markets, the share of buyers in market 1 in a quasi-equilibrium can be the slightly smaller number $\frac{1-\gamma}{4}$.

Proposition 10 has the advantage of applying for any distribution of valuations, but it has several limitations relative to our analysis of uniformly-distributed valuations: It provides sufficient conditions for the existence of quasi-equilibria but not necessary ones; it doesn't describe the equilibrium set for small numbers of buyers and sellers; and it doesn't tell us when the requirement that equilibrium involves whole numbers in each market can be satisfied. The only way we know of to provide that sort of more detailed characterization is to specify a particular distribution for buyer values and perform the analogs of the explicit calculations of Section 4.

7. Price Competition

The auction sites in our base model are not competing in price; we have implicitly assumed they set a price of zero. Our results also describe what happens when competing auction sites charge any common price that is not so large as to drive away buyers or sellers. This is a reasonable approximation to many of the real-world examples discussed in the introduction. Christie's and Sotheby's were recently convicted of price-fixing in connection with their joint adherence to a complex schedule of buyers' and sellers' commissions.[16] Amazon also matched eBay's multi-tiered schedule of listing fees and commissions when it entered. In other cases, however, there has been price competition: Yahoo! Auctions initially charged no listing fees, and Phillips did not immediately match Sotheby's and Christie's 1975 price increase.[17] In this section, we examine an extension of our base model with one extra stage: Before the sellers and buyers choose between sites, the sites simultaneously announce the listing fees that they will charge sellers.

As in models of firms' competing to sell products that provide network externalities, little can be said about prices or market shares in this model without some extra assumptions about how the equilibrium multiplicity is resolved in the game between the buyers and sellers that follows the sites' setting prices.

PROPOSITION 11: *Consider a three-stage game in which the auction sites first set listing fees and buyers and sellers then choose locations as in the model of Section 2.*

[16] The firms were also accused of price-fixing in 1975 when they simultaneously changed their commission schedules, instituting a 10% "buyer's premium."

[17] See Learmount (1985), p. 162.

(a) For any $p \in [0, \overline{p}(S, B)]$ *the model has an equilibrium in which auction site 1 charges a listing fee of p and attracts all buyers and all sellers.*

(b) If the model of Section 2 has an equilibrium with S_1 *sellers and* B_1 *buyers at site 1, then for any* $p \in [0, \min\{\overline{p}(S_1, B_1), \overline{p}(S - S_1, B - B_1)\}]$ *the price-setting game has an equilibrium in which both sites charge a listing fee of p and* S_1 *sellers and* B_1 *buyers choose site 1.*

Proof: For part (a) suppose that the equilibrium selection rule in the second stage is that buyers and sellers all go to auction site 1 whenever site 1 sets a price of *p* and that all buyers and sellers go to auction site 2 whenever site 1 sets any other price. For part (b) assume that the selection rule is that the S_1 sellers and B_1 buyers choose market 1 whenever market 1 if both sites set a price of *p*, and that if one site deviates to a price other then *p* then no buyers or sellers go to it. QED

The most common equilibrium refinement in the duopoly network externality literature is to assume that consumers coordinate on the equilibrium that is Pareto optimal for them. In that literature, it leads to Bertrand-like competition, and implies that all consumers will buy from one firm. Such a strong refinement does not seem reasonable in the competing auctions context: Yahoo! did not attract all buyers and sellers despite substantially undercutting eBay. Moreover, in this setting the refinement need not eliminate equilibria with two active markets. For example, in the model with 5 sellers, 10 buyers, and uniformly distributed valuations we saw that the equilibrium set consisted of all splits with 0, 1, 2, 3, 4, or 5 sellers and twice as many buyers as sellers in market 1. These equilibria are not Pareto-ranked: buyers are best off in the smallest market and sellers in the largest market. Hence, even with a Pareto optimality refinement one could support equilibria with zero listing fees and any split of the sellers by assuming that any change in the listing fee would cause all buyers and sellers to switch to the other auction site.

Caillaud and Jullien (2003) note that an alternate way to shrink the equilibrium set is to allow firms to use more sophisticated pricing instruments. This would dramatically alter the outcome in our model as well. As Schwartz and Ungo (2002) note, an equilibrium with two active markets always yields less expected surplus than an equilibrium with all buyers and sellers at a single auction site. Hence, given any strategy profile in which both sites are active, either site could deviate and make individualized offers to each buyer and seller, promising to pay them the positive or negative amount that would make them ε better off than they would be under the original strategy profile if all buyers and sellers come are attracted by that site, and "guaranteeing" that the site will attract all buyers and sellers by promising to make a very large payment to each of them if not all buyers and sellers come. The fact that attracting all buyers and sellers to a single site maximizes total surplus implies that this deviation produces greater profits (assuming both sites profits were nonnegative in the original profile).

8. Thin Markets

We now test the robustness of the conclusion that auction sites of quite different sizes can coexist to the possibility that markets are "thin," in the sense of there being very few items available for trade. Specifically, we specialize again to uniformly distributed values and suppose that each seller only has the good with a probability $q << 1$. We suppose that buyers and sellers choose markets before either the buyer's uncertainty or the seller's is resolved; we think of q as the probability that the seller has a good of the appropriate type to sell in the "current period."

The seller's expected utility when he attends a market with S other sellers and B buyers, and when he has a good to sell is $1-\frac{qS+1}{B+1}$. If qS is sufficiently small ($qS < 4/B^2$ will do), sellers will never be willing to go to site 1 if $B_1 < B_2$. This is intuitive: when q is very small, each seller expects to be a monopolist at either site, and so prefers to be at the site with more buyers.

However, when $qS \to 0$ there are vanishingly few objects offered for sale, so this is a fairly extreme version of a thin market. One less extreme version of thinness would be to consider a model with exactly three sellers, each of whom has an object to sell (so we go back to $q=1$.) Here, there would be equilibria with two active markets even when the number of buyers is very large. Specifically, one can show using the payoff functions given in Section 4 that $S_1 = 1$, $S_2 = 2$, $B_1 = (2B-1)/5$ and $B_2 = (3B+1)/5$ is an equilibrium of the model with uniformly distributed values whenever $2B-1$ is a multiple of five.

9. Concluding Comments

We would like to develop a model that incorporates adverse selection in the market-participation decision. Our casual empiricism suggests that a major reason that the Amazon and Yahoo auction sites have struggled is that they tried to compete by having zero listing fees. This led to their listings being filled up with products being offered by non-serious sellers with very high reserve prices. If we suppose that there is a cost to reading web pages, or to investigating the quality of a good and/or its seller, then buyers will prefer to frequent sites with a high percentage of “good” listings- listings by reputable sellers who have high-quality goods and are willing to sell them at a reasonable price. In this case, a market with too many “bad” sellers might collapse. However, two markets might be able to co-exist if sites have some background flow of captive traffic from people who click in from Yahoo or Amazon without considering another auction site.

The issue of reserve prices poses a problem for a would-be new market site: On the one hand, when the market is new, sellers may not expect to get competitive bids, and so be unwilling to participate unless they can protect themselves with a reserve price. However, while the imposition of a uniform reserve price in a market can increase the payoff of sellers for any fixed buyer-seller ratio, it lowers the overall efficiency of the market, and so we would expect it to reduce the viability of the new market.

We would like to point out that it is not necessary to assume that both buyers and sellers recognize that they have a market impact to obtain our conclusions; it is sufficient that one side does. (This is a consequence of Lemma A2.)

Finally we would like to note that while the paper has analyzed competition between two markets, its analysis also applies to the study of $2M$ markets, M “smaller” and M “larger”. Such a configuration will be an equilibrium provided that it is an equilibrium for $M = 1$. This shows that any tendency to have only two markets, as opposed to more, must be due either to “relatively small” numbers of participants, or to agglomerative forces not captured by our model. One reason why such configurations may be less common in practice is that our model suggests they could be quite fragile –if two or more of the markets merge, the merged entity may be sufficiently large relative to the others so as to attract all of the patrons of every small market.

Appendix

Proof of Proposition 4: The "only if" direction is trivial. To prove the "if" direction we use a lemma that shows that, under three regularity conditions, it suffices to work with two of the four incentive constraints.

LEMMA A1: Fix S and B with $S+1<B$. Consider a general model in which S sellers and B buyers simultaneously choose between two locations, and receive payoffs of $u_s(S_i,B_i)$ and $u_b(S_i,B_i)$ if they choose market i and market i attracts S_i sellers and B_i buyers. Assume $B_1 \le B/2$ and that the utility functions satisfy three conditions:

(A1-Boundary) $\begin{array}{l} u_s(S,B) > 0 \text{ if } B>S,\ u_s(S,B)=0 \text{ if } B \le S \\ u_b(S,B) > 0 \text{ if } S>0,\ u_b(0,B)=0 \end{array}$.

(A2-Monotonicity) If $B>S$, then $\frac{\partial u_s}{\partial S}<0, \frac{\partial u_s}{\partial B}>0, \frac{\partial u_b}{\partial S}>0,$ and $\frac{\partial u_b}{\partial B}<0$.

(A3-Large Market Efficiency) If $B_1<B_2$ then $u_s(S_1,B_1)<u_s(S_2,B_2)$ or $u_b(S_1,B_1)<u_b(S_2,B_2)$.

Then, there exists an S_1 such that $(S_1, S-S_1, B_1, B-B_1)$ is a quasi-equilibrium if and only if there exists an S_1 such that $(S_1, S-S_1, B_1, B-B_1)$ satisfies the (B1) and (S1) constraints.

A proof of this lemma is given in Ellison and Fudenberg (2002). Here is a sketch: If one fixes an allocation S_1, B_1 satisfying (S1) and (B1), then large market efficiency implies that either the buyers or the sellers (or both) are getting higher utility in market 2. If both, then (S2) and (B2) are obviously satisfied and we are done. If only (B2) is satisfied, then the allocation we started with is not a quasi-equilibrium. In this case, however, we can add sellers to market 1 until (S2) is just satisfied. At this allocation sellers get higher utility in the small market. The other three constraints therefore hold: (B1) continues to hold because we've made market 1 more attractive; (S1) holds because sellers are doing better in market 1; and (B2) holds because buyers are doing better in market 2 (which follows from large market efficiency). The case when only (S2) holds is similar.

It is clear from inspection that the utility functions for the model with uniform valuations satisfy (A1) and (A2). For (A3) note that we can rewrite the buyer utility as $u_b = \frac{(1-\overline{p})}{2}(1-\overline{p}(1+\frac{1}{B}))$;[18] thus if prices are higher in market 1 and B_1 is smaller than B_2, then buyers must be better off in market 2. Hence, Lemma A1 applies and we need only determine the range of values of B_1 for which the (S1) and (B1) constraints can be simultaneously satisfied.

The (S1) constraint can be rewritten as $S_1 + 1 \le c(B_1 + 1)$, where $c = \frac{S+3}{B+2}$. For a given B_1, this holds for all S_1 below the line where the constraint holds with equality. The monotonicity of the buyer's utility function implies that if the (B1) constraint holds for a given S_1, then it holds for all larger S_1. Hence, there is an S_1 satisfying both (S1) and (B1) if and only if (B1) is satisfied when (S1) holds with equality.

When (S1) holds with equality, the (B1) constraint, which is $\frac{S_1(1+S_1)}{B_1(1+B_1)} \ge \frac{S_2(1+S_2)}{(B_2+1)(B_2+2)}$, becomes $\frac{c(B-B_1+1)(B-B_1+2)}{B_1} \ge \frac{(S-S_1)(S-S_1+1)}{S_1}$.

Further algebra shows that if we define $z = \frac{B_1+1}{B+2}$ and $f = \frac{1}{B+2}$, this can be rewritten as

$$c^2z^3 - \left(cf + c^2(2+f)\right)z^2 + \left((cf+c^2)(1+f)+cf\right)z - cf(1+f) \geq$$
$$c^2z^3 + \left(-(cf+c-2f)c - c(c-f)\right)z^2 + \left((c-2f)cf + (cf+c-2f)(c-f)\right)z -$$
$$f(c-2f)(c-f)$$

Subtracting the right-hand side from the left yields the quadratic equation

$$-4cz^2 + (5c - c^2 + 4cf - 2f)z + 2f^2 - c - 4cf + c^2 \geq 0.$$

Rearranging terms gives

(a") $\left[-4cz^2 + (5c-c^2)z - c + c^2\right] + \left[2f(f + 2cz - z - 2c)\right] \geq 0.$

An exact bound on the quasi-equilibrium set is obtained by letting z be the smallest value for which (a") holds with equality and setting $\underline{B}_1 = z(B+2)-1$. The fact that (B1) is satisfied at $(B_1, S_1) = (B/2, S/2+1/2)$ (where (S1) holds with equality) implies that the expression (a") is positive at $z = 1/2$. Hence, the smallest z satisfying (a") is just the smaller root of the quadratic equation obtained by setting the left-hand side of the inequality equal to zero. An explicit formula for $\underline{B}_1$ is therefore easy to obtain.

To obtain the approximations to $\underline{B}_1$ given in the Proposition, we note that writing $Q(z)$ for the term in the first set of square brackets in (a") and $L(z,f)$ for the remainder of the left-hand side, the quadratic is of the form $Q(z) + L(z,f) = 0$ with $-f(1+4c) < L(z,f) < 0$ for $z \in (0,1/2)$. (The latter inequality follows from $f = \frac{c}{S+3} = \frac{1}{B+2} < \frac{S+3}{4(B+2)} = \frac{c}{4}$.) Hence, the smaller root of $Q(z) + L(z,f) = 0$ lies between the smaller root of $Q(z) = 0$ and the smaller root of $Q(z) - f(1+4c) = 0$. Now $Q(z) = 4c(1-z)(z - \frac{1-c}{4})$, so the smaller root of $Q(z) = 0$ is $z^* = (1-c)/4$. This value of z^* corresponds with $\frac{\underline{B}_1}{B} = \frac{1}{4}(1 - \frac{S}{B}) - \frac{5}{4B}$, which proves that

[18] To show this, note that S/B = (S+1)/(B+1) + (S/B - (S+1)/(B+1)).

$\frac{B_1}{B} > \frac{1}{4}\left(1-\frac{S}{B}\right)-\frac{5}{4B}$. To prove the other required inequality, note that a consequence of the factorization of $Q(z)$ is that $Q(z) > 2c(z-\frac{1-c}{4})$ when $z < 1/2$. This implies that $Q(z^* + f(1+4c)/2c) - f(1+4c) > 0$, so the smaller root of $Q(z) - f(1+4c) = 0$ satisfies $z < z^* + f(1+4c)/2c$. This gives $\frac{B_1}{B} < \frac{1}{4}\left(1-\frac{S}{B}\right)-\frac{5}{4B}+\frac{(B+2)}{2B(S+3)}+\frac{2}{B} < \frac{1}{4}\left(1-\frac{S}{B}\right)+\frac{3}{4B}+\frac{1}{S}$.

QED

LEMMA A2: Suppose buyers' values have the uniform distribution. Fix *B* and *S* with $B > S+2$. For every partition (B_1, B_2) with $\frac{B_1}{B} \in [\frac{1}{2}-\frac{S}{2B}, \frac{1}{2}+\frac{S}{2B}]$, there is a quasi-equilibrium (S_1, S_2, B_1, B_2) with $u_s(S_1, B_1) = u_s(S_2, B_2)$. Specifically, choosing S_1 and S_2 with $\frac{S_1+1}{B_1+1} = \frac{S_2+1}{B_2+1}$ gives such a quasi-equilibrium.

Proof of Lemma A2: When expected prices are equal, both seller constraints are satisfied. Equal prices also imply that $\gamma = \frac{S_i+1}{B_i+1}$ is the same in both markets, so $(S_1+1)(B_2+1) = (S_2+1)(B_1+1)$. Then by canceling terms equal to γ we can rewrite the buyer constraints as

$$(a') \quad \frac{S_1}{B_1} \geq \frac{S_2}{(B_2+2)}; \quad (b') \quad \frac{S_2}{B_2} \geq \frac{S_1}{(B_1+2)}.$$

Rewrite (*a'*) and (*b'*) as

$S_i(B_j+2) \geq S_j B_i$, and add and subtract terms to obtain

$(S_i+1)(B_j+1)+S_i-(B_j+1) \geq (S_j+1)(B_i+1)-(S_j+B_i+1)$

Divide both sides by $(B_i+1)(B_j+1)$

$$(*) \qquad \frac{S_i+1}{B_i+1} \geq \frac{S_j+1}{B_j+1} + \frac{B_j-(S_i+S_j+B_i)}{(B_i+1)(B_j+1)}.$$

Using the fact that prices are equal, this is equivalent to $\frac{B_i}{B} \leq \frac{1}{2}+\frac{S}{2B}$ or

$$(**) \qquad \frac{B_i}{B} \in [\frac{1}{2}-\frac{S}{2B}, \frac{1}{2}+\frac{S}{2B}].$$

For any B_1, B_2 that satisfy (**), the buyer constraints are satisfied for the S_1 and S_2 that equate the expected prices in the two markets. The last step of the proof is to show that under (**) there must exist a pair S_1, S_2 that does equate the expected prices. Holding B_1, B_2 fixed, and setting $S_2 = S - S_1$, the difference in expected prices is

$$\bar{p}(S_1,B_1)-\bar{p}(S_2,B_2) = \frac{(B_1-S_1)(B_2+1)-(B_2-(S-S_1))(B_1+1)}{(B_1+1)(B_2+1)} = \frac{(B_1+1)(S-S_1)-(B_2+1)S_1+(B_1-B_2)}{(B_1+1)(B_2+1)}$$

which is a linearly decreasing function of S_1. When $S_1 = 0$ the difference is proportional to $S(B_1+1)+(B_1-B_2)$; from (**) this is at least $SB_1 > 0$. Similarly when $S_1 = S$ the difference is proportional to $-S(B_2+1)+(B_1-B_2) < -SB_2 < 0$. So there is a solution with $0 < S_1, S_2 < S$.

QED

Proof of Proposition 5: We will first construct an equal-price partition $(B_1, B_2, \hat{S}_1^j, \hat{S}_2^j)$ that approximates the target ratios, but where only B_1 and B_2 are guaranteed to be integers; we will then use this partition to construct an integer-valued partition (B_1, B_2, S_1^*, S_2^*) where all of the incentive constraints are satisfied but prices are only approximately equal.

Assume that $\alpha < 1/2$, let $\gamma^* = \frac{\lfloor \gamma(B+2) \rfloor}{B+2}$, and $\alpha^* = \frac{\lceil \alpha(B+2) \rceil}{B+2}$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x, and $\lceil x \rceil$ is the smallest integer greater than or equal to x.[19] Note that $\gamma^* \le \gamma$,and $1-2\alpha^* \le 1-2\alpha$; since we have already assumed that $\gamma > 1-2\alpha$, we know that $\gamma^* > 1-2\alpha^*$ for $B > \frac{3}{\gamma-(1-2\alpha)}$. Note also that for B sufficiently large we have $\alpha^* < 1/2$.

Let $v^* = \min\{\alpha^*, 1-2\alpha^*\}$, and let $k = \left\lceil \frac{1}{v^*} \right\rceil$. Define $B_1 = \alpha^*(B+2)-1$, $B_2 = B - B_1$. For any non-negative integer *j,* define $\gamma^j = \gamma^* + \frac{2j}{B+2}$ and $\tilde{\gamma}^j = \gamma^* + \frac{j}{B+2}$. If $\alpha^* \ge 1-2\alpha^* = v^*$, set $S^j = \gamma^j(B+2)-2$, $\hat{S}_1^j = \gamma^j(B_1+1)-1$, and $\hat{S}_2^j = \gamma^j(B_2+1)-1$. If $\alpha^* < 1-2\alpha^*$, define $S^j = \tilde{\gamma}^j(B+2)-2$, $\hat{S}_1^j = \tilde{\gamma}^j(B_1+1)-1$, and $\hat{S}_2^j = \tilde{\gamma}^j(B_2+1)-1$. In either case, by construction B_1, B_2 and S^j are integers, $\hat{S}_1^j + \hat{S}_2^j = S^j$, and $\frac{\hat{S}_1^j+1}{B_1+1} = \frac{\hat{S}_2^j+1}{B_2+1}$. If $\alpha^* \ge 1-2\alpha^* = v^*$, then

$\frac{S^j}{B} = \frac{\gamma^j(B+2)-2}{B} = \frac{\gamma^*(B+2)-2}{B} + \frac{2j}{B} \in [\gamma^* - \frac{2(1-\gamma^*)}{B}, \gamma^* + \frac{2(k-(1-\gamma^*))}{B}]$, which is within ε of γ if $B > \frac{2k}{\varepsilon}$. If $\alpha^* < 1-2\alpha^*$, then a similar calculation shows

$\frac{S^j}{B} \in [\gamma^* - \frac{1(1-\gamma^*)}{B}, \gamma^* + \frac{k-(1-\gamma^*)}{B}]$; this is also within ε of γ if $B > \frac{2k}{\varepsilon}$.

[19] The case $\alpha > 1/2$ is symmetric. A separate argument is needed for $\alpha = 1/2$; we omit this argument here but will provide it on request.

Note also that $\hat{S}_1^{j+1} - \hat{S}_1^j = \gamma^{j+1}(B_1 + 1) - \gamma^j (B_1 + 1) = (\gamma^{j+1} - \gamma^j)\alpha^*(B+2) = 2\alpha^*$ if $\alpha^* \geq 1 - 2\alpha^*$, and $\hat{S}_1^{j+1} - \hat{S}_1^j = \tilde{\gamma}^{j+1}(B_1 + 1) - \tilde{\gamma}^j (B_1 + 1) = \alpha^*$ if $\alpha^* < 1 - 2\alpha^*$.

The assumption that $\alpha \in \left(\frac{1}{2} - \frac{\gamma}{2}, \frac{1}{2}\right)$ and the fact that γ^j and $\tilde{\gamma}^j$ are each larger than γ imply that for B large enough, $\frac{B_1}{B} = \frac{\lceil \alpha^*(B+2) \rceil - 1}{B} \in (\frac{1}{2} - \frac{\gamma}{2}, \frac{1}{2}) \subset [\frac{1}{2} - \frac{\gamma^j}{2}, \frac{1}{2}]$. Thus Lemma A2 implies that each partition $(B_1, B_2, \hat{S}_1^j, \hat{S}_2^j)$ satisfies all four incentive constraints. If $\hat{S}_1^j$ is an integer for any $j \in \{0, 1, ..., k\}$ we are done. If not and if $\alpha^* \geq 1 - 2\alpha^* = \nu^*$, let m be the smallest integer j with $\lfloor \hat{S}_1^j \rfloor = \lfloor \hat{S}_1^{j-1} \rfloor$. We know that $\hat{S}_1^m - \lfloor \hat{S}_1^m \rfloor \geq 2\alpha^*$, so $\lceil \hat{S}_1^m \rceil - \hat{S}_1^m \leq 1 - 2\alpha^* = \nu^*$. If $\alpha^* < 1 - 2\alpha^*$, let m be the largest j integer with $\hat{S}_1^j < \lceil \hat{S}_1^0 \rceil$, then $\lceil \hat{S}_1^m \rceil - \hat{S}_1^m \leq \alpha^* = \nu^*$. In either case let $S_1^* = \lceil \hat{S}_1^m \rceil$ and $S_2^* = S - S_1^*$. We will now show that (B_1, B_2, S_1^*, S_2^*) is an equilibrium; that is, the deviation of the partition from exactly equal prices (which is necessary to satisfy the integer constraint) is small enough that the incentive constraints are still satisfied.

Since we set the number of sellers in market 1 to be slightly higher than the number needed for equal prices, the constraints (B1) – that buyers are willing to stay in market 1 – and (S2) – that sellers stay in market 2 – will be the easiest to check. For (B1), note that by Lemma A2, $(B_1, B_2, \hat{S}_1^m, \hat{S}_2^m)$ satisfies the constraint; the fact that $S_1^* > \hat{S}_1^m$ implies it is satisfied by (B_1, B_2, S_1^*, S_2^*). For (S2), note that note that since $(B_1, B_2, \hat{S}_1^m, \hat{S}_2^m)$ has equal prices, the fact that $S_1^* > \hat{S}_1^m$ implies $\frac{S_2^* + 1}{B_2 + 1} < \frac{\hat{S}_2^m + 1}{B_2 + 1} = \frac{\hat{S}_1^m + 1}{B_1 + 1} < \frac{S_1^* + 1}{B_1 + 1} < \frac{S_1^* + 2}{B_1 + 1}$. Therefore a seller gets a better expected price by staying in market 2.

The constraint that sellers are willing to stay in market 1 requires that

$\frac{\hat{S}_1^m+1}{B_1+1}\frac{S_1^*+1}{\hat{S}_1^m+1} \le \frac{S_2^*+2}{B_2+1} = \frac{\hat{S}_2^m+1}{B_2+1}\frac{S_2^*+2}{\hat{S}_2^m+1}$ or, using the facts that prices are equal at $(B_1, B_2, \hat{S}_1^m, \hat{S}_2^m)$, $\frac{S_1^*+1}{\hat{S}_1^m+1} \le \frac{S_2^*+2}{\hat{S}_2^m+1}$. Since $S_1^* - \hat{S}_1^m = \hat{S}_2^m - S_2^*$, we can rewrite this as $\frac{S_1^* - \hat{S}_1^m}{\hat{S}_1^m+1} \le \frac{\hat{S}_1^m - S_1^*+1}{\hat{S}_2^m+1}$, or $S_1^* - \hat{S}_1^m \le \frac{1+\hat{S}_1^m}{2+S}$. By construction $S_1^* - \hat{S}_1^m \le \nu^*$, and $\frac{1+\hat{S}_1^m}{2+S} = \frac{\gamma^m \alpha^* (B+2)}{2+S} = \alpha^* \frac{\gamma^* (B+2)+2m}{2+S} \ge \alpha^*$. Finally we come to the constraint that buyers be willing to stay in market 2. This is $\frac{S_2^*(S_2^*+1)}{B_2(B_2+1)} \ge \frac{S^*_1(S_1^*+1)}{(B_1+1)(B_1+2)}$, which we can write as $\frac{S_2^*}{B_2}\left(\frac{S_2^*+1}{B_2+1}\frac{B_1+1}{S_1^*+1}\right) \ge \frac{S_1^*}{B_1+2}$. When we ignored the integer constraint, the term in the brackets was equal to 1; the issue now is whether the integer partition keeps this term close enough to 1.

Note that $\frac{S_2^*+1}{B_2+1} = \frac{\hat{S}_2^m+1}{B_2+1}\frac{S_2^*+1}{\hat{S}_2^m+1} \ge \frac{\hat{S}_2^m+1}{B_2+1}\left(1-\frac{\nu^*}{\hat{S}_2^m+1}\right)$.

Also $\frac{B_1+1}{S_1^*+1} = \frac{B_1+1}{\hat{S}_1^m+1}\frac{\hat{S}_1^m+1}{S_1^*+1} \ge \frac{B_1+1}{\hat{S}_1^m+1}(1-\frac{\nu^*}{S_1^*+1})$. Using the fact that $\frac{\hat{S}_2^m+1}{B_2+1} = \frac{\hat{S}_1^m+1}{B_1+1}$, we conclude that the incentive constraint is satisfied if $\frac{S_2^*}{B_2}\left(1-\frac{\nu^*}{\hat{S}_2^m+1}\right)\left(1-\frac{\nu^*}{\hat{S}_1^*+1}\right) \ge \frac{S_1^*}{B_1+2}$. If we rewrite this as $\frac{S_2^*}{B_2}(1-x)(1-y) \ge \frac{S_1^*}{B_1+2}$, where x and y are defined as the two fractions inside of the large brackets, we see that a sufficient condition for incentive compatibility for large B is $\frac{S_2^*}{B_2}(1-\delta) > \frac{S_1^*}{B_1+2}$ for $\delta = \frac{\nu^*}{\hat{S}_2^m+1} + \frac{\nu^*}{S_1^*+1}$. Moreover,

(*) $\delta = \dfrac{\nu^*}{(1-\alpha)\gamma\alpha(B+2)} + O(1/B^2)$.

By algebra similar to the proof of Lemma A2, the condition $\dfrac{S_2^*}{B_2}(1-\delta) \geq \dfrac{S_1^*}{B_1+2}$ is equivalent

to

(**) $\dfrac{S_2^*+1}{B_2+1} - \dfrac{S_1^*+1}{B_1+1} \geq \delta\left(\dfrac{S_2^*+1}{B_2+1}\right) + \dfrac{B_1 - S_2^* + 1}{(B_1+1)(B_2+1)}(1-\delta) - \dfrac{S_1^* + B_2 + 1}{(B_1+1)(B_2+1)}$.

We claim that the left hand side is at least $-\dfrac{\nu^*}{\alpha(1-\alpha)(B+2)}$, because

$$\frac{S_2^*+1}{B_2+1} - \frac{S_1^*+1}{B_1+1} = \frac{S_2^*+1}{B_2+1} - \frac{\hat{S}_2^m+1}{B_2+1} + \frac{\hat{S}_1^m+1}{B_1+1} - \frac{S_1^*+1}{B_1+1} \geq -\nu^*\left(\frac{1}{B_2+1} + \frac{1}{B_1+1}\right) \geq -\frac{\nu^*}{\alpha(1-\alpha)(B+2)}.$$

We also know that the term $\left(\dfrac{S_2^*+1}{B_2+1}\right)$ in the right-hand side of (**) equals $\gamma + O(1/B)$.

Therefore it will be sufficient to show that

$-\dfrac{\nu^*}{\alpha(1-\alpha)(B+2)} \geq \delta\gamma + \dfrac{B_1 - S_2^* + 1}{(B_1+1)(B_2+1)}(1-\delta) - \dfrac{S_1^* + B_2 + 1}{(B_1+1)(B_2+1)} + O\left(1/B^2\right)$. (The $O\left(1/B^2\right)$ error term arises because δ is order $1/B$.)

Substituting the approximation for δ from (*) into the expression $\delta\gamma$, and using the fact that $\dfrac{\delta}{B} = O(1/B^2)$ to replace $1-\delta$ by 1, it is sufficient to show that

$$\frac{2\nu^*}{\alpha(1-\alpha)(B^*+2)} \leq \frac{S_1^* + B_2 + 1}{(B_1+1)(B_2+1)} - \frac{B_1 - S_2^* + 1}{(B_1+1)(B_2+1)} + O(1/B^2).$$

The right hand side of this expression is

$$\frac{S_1^*+1}{(B_1+1)(B_2+1)}+\frac{B_2+1}{(B_1+1)(B_2+1)}-\frac{B_1+1}{(B_1+1)(B_2+1)}+\frac{S_2^*+1}{(B_1+1)(B_2+1)}+O(1/B^2)=$$

$$\frac{\gamma}{(1-\alpha)(B+2)}+\frac{1}{\alpha(B+2)}-\frac{1}{(1-\alpha)(B+2)}+\frac{\gamma}{\alpha(B+2)}+O(1/B^2)\,.$$

Multiplying through by $\alpha(1-\alpha)B$ and collecting terms, we see that the constraint is satisfied if $2v^* \le \gamma+1-2\alpha+O(1/B)$. Since $v^* \le 1-2\alpha$ and $v^* < \gamma$, we conclude that the incentive constraint is satisfied for all sufficiently large *B*. QED

Proof of Proposition 7: One way to generate a sample Y^M of *mB* draws from *F* is to first generate a sample Y^N of *nB* i.i.d. draws, and then randomly select a subset of *mB* elements; we will use this method to relate the distributions of the order statistics of the two samples. Let q_i be the probability that the *i*th highest draw from Y^N is one of the *mS* highest elements of Y^M. This probability is independent of the realized values of the order statistics; it depends only on which elements of Y^N are chosen. In particular, for $i=1$ to *mS*, q_i is simply the probability that the element in question is chosen, namely $\frac{m}{n}$; for *i* = *mS*+1 and thereafter each subsequent q_i is strictly less than the preceding one since this $i = mS + j$ will only be one of the *mS* highest elements if it is chosen and at least *j* of the higher realizations are not.

Then

$$w(mS,mB)=E(v\mid v\ge v^{mS:mB})=\frac{\sum_{i=1}^{nB} q_i E(v^{i:nB})}{mS},$$

while

$$w(nS,nB)=E(v\mid v\ge v^{nS:nB})=\frac{\sum_{i=1}^{nS} E(v^{i:nB})}{nS}=\frac{\sum_{i=1}^{nB} Q_i E(v^{i:nB})}{nS},$$

where Q_i is an indicator function that equals 1 for $i =$ 1 to nS and 0 otherwise.

So

$$w(mS, mB) - w(nS, nB) = E(v \mid v \geq v^{mS:mB}) - E(v \mid v \geq v^{nS:nB})$$

$$= \frac{\sum_{i=1}^{nB} q_i E(v^{i:nB})}{mS} - \frac{\sum_{i=1}^{nB} Q_i E(v^{i:nB})}{nS} = \sum_{i=1}^{nB} c_i E(v^{i:nB})$$

where $c_i = \dfrac{nq_i - mQ_i}{nmS}$.

Now for $i =$ 1 to mS, $c_i = 0$, for $i = mS$+1 to nS, c_i is negative, and for $i > nS$ c_i is positive, and $\sum_{i=1}^{nB} c_i = 0$. Since the $E(v^{i:nB})$ are monotone decreasing in i, it follows that $\sum_{i=1}^{nB} c_i E(v^{i:nB}) < 0$. QED

Proof of Proposition 8: We start with the assumption that B and $S = \gamma B$ are positive integers and later consider a continuous extension of buyers' and sellers' utility functions. We will calculate the seller utility $u_s(\gamma B, B)$ and total surplus per seller $w(\gamma B, B)$, which allows us to derive the buyer utility through the equation:

$$u_b(\gamma B, B) = \gamma \left[w(\gamma B, B) - u_s(\gamma B, B) \right].$$

Seller utility and total surplus per seller can be calculated as

$$V_G(\gamma B, B) = \int_0^{\bar{b}} G(v) f^{\gamma B+1:B}(v) dv,$$

where $f^{\gamma B+1:B}(v)=\binom{B}{\gamma B}(B-\gamma B)F(v)^{B-\gamma B-1}\left(1-F(v)\right)^{\gamma B}f(v)$ is the probability density

function of $v^{\gamma B+1:B}$, and $G(v)=v$ for seller utility and $G(v)=\frac{\int_v^{\bar{b}} xf(x)dx}{1-F(v)}$ for total surplus.

Choose an ε-neighborhood of $\bar{v}=F^{-1}(1-\gamma)$ which lies strictly inside the support of the distribution function F.[20] We can write the integral $V_G(\gamma B,B)$ as the sum of two integrals:

$$\gamma V_G(\gamma B,B)=\int_{\bar{v}-\varepsilon}^{\bar{v}+\varepsilon} G(v)f^{\gamma B+1:B}(v)dv+\int_{[0,\bar{b}]\setminus[\bar{v}-\varepsilon,\bar{v}+\varepsilon]} G(v)f^{\gamma B+1:B}(v)dv=J_1+J_2.$$

We know that the derivatives of *f* are uniformly bounded on the first interval, so we will be able to use a Taylor approximation to approximate the integral J_1. Before doing so, we will use the fact that the distribution of the order statistic is converging exponentially fast to bound J_2.

To bound J_2 we first define the function $\hat{G}(v)=\left(1-F(v)\right)G(v)$. This function is four times continuously differentiable, and satisfies $0\le\hat{G}(v)\le\int_0^{\bar{b}} xf(x)dx=Ev$, which is finite. We rewrite J_2 substituting $\hat{G}$ for *G*, multiplying and dividing by S/B, and making the change of variables $y=F(v)$:

$$J_2=\frac{1}{\gamma}\int_{[0,1]\setminus[F(\bar{v}-\varepsilon),F(\bar{v}+\varepsilon)]}\hat{G}(F^{-1}(y))\binom{B-1}{\gamma B-1}(B-\gamma B)y^{B-\gamma B-1}(1-y)^{\gamma B-1}dy$$

[20] Such a neighborhood exists because F is strictly monotone within its support and $0<\gamma<1$.

Write $r^{k:N}(y)$ for the probability density function of the k^{th} highest of N draws from a uniform distribution, $r^{k:N}\left(y\right) \equiv \binom{N}{k-1}\left(N-k\right) y^{N-k}\left(1-y\right)^{k-1}$. Note that the expression for J_2 is

$J_2 = \frac{1}{\gamma} \int_{[0,1]\setminus\left[F\left(\bar{v}-\varepsilon\right),F\left(\bar{v}+\varepsilon\right)\right]} \hat{G}(F^{-1}\left(y\right))r^{\gamma B:B-1}(y)dy$. Note also that

$y^{B-\gamma B-1}\left(1-y\right)^{\gamma B-1} = \left[y^{1-\gamma}\left(1-y\right)^{\gamma}\right]^{B}/(y(1-y))$ is single-peaked with its maximum at $(B-\gamma B-1)/(B-2)$. The function $a(y) = y^{1-\gamma}\left(1-y\right)^{\gamma}$ reaches its maximum at $1-\gamma$. For B sufficiently large $r^{\gamma B:B-1}(y)$ has its maximum in the interval $(F(\bar{v}-\varepsilon/4), F(\bar{v}+\varepsilon/4))$. The fact that $\int_0^1 r^{\gamma B:B-1}(y)dy = 1$ (or more generally, is finite) then implies that there exists an M_1 with

$$\max\left[r^{\gamma B:B-1}\left(F\left(\bar{v}-\varepsilon/2\right)\right), r^{\gamma B:B-1}\left(F\left(\bar{v}+\varepsilon/2\right)\right)\right] < M_1$$

for all sufficiently large B. We can choose M_2 such that

$\max\left[\frac{F(\bar{v}-\varepsilon/2)(1-F(\bar{v}-\varepsilon/2))}{F(\bar{v}-\varepsilon)(1-F(\bar{v}-\varepsilon))}, \frac{F(\bar{v}+\varepsilon/2)(1-F(\bar{v}+\varepsilon/2))}{F(\bar{v}+\varepsilon)(1-F(\bar{v}+\varepsilon))}\right] < M_2$. We can now conclude that

$$\max_{y\in[0,1]\setminus\left[F\left(\bar{v}-\varepsilon\right),F\left(\bar{v}+\varepsilon\right)\right]} r^{\gamma B:B-1}\left(y\right) \leq M_1 M_2 \max\left[\left(\frac{a\left(F\left(\bar{v}-\varepsilon\right)\right)}{a\left(F\left(\bar{v}-\varepsilon/2\right)\right)}\right)^{B}, \left(\frac{a\left(F\left(\bar{v}+\varepsilon\right)\right)}{a\left(F\left(\bar{v}+\varepsilon/2\right)\right)}\right)^{B}\right].$$

Note that $C = \max\left[\frac{a\left(F\left(\bar{v}-\varepsilon\right)\right)}{a\left(F\left(\bar{v}-\varepsilon/2\right)\right)}, \frac{a\left(F\left(\bar{v}+\varepsilon\right)\right)}{a\left(F\left(\bar{v}+\varepsilon/2\right)\right)}\right] < 1$ which allows us to bound the right-hand side of this expression by $M_1 M_2 C^B$ from above. Therefore, we find that

$$J_2 \leq \frac{1}{\gamma} Ev \max_{y\in[0,1]\setminus\left[F\left(\bar{v}-\varepsilon\right),F\left(\bar{v}+\varepsilon\right)\right]} r^{\gamma B:B-1}(y) \leq \frac{1}{\gamma} Ev M_1 M_2 C^B,$$

which is certainly $o\left(\frac{1}{B}\right)$.

The bound on J_2 implies that

$$V_G\left(\gamma B, B\right) = \int_{\bar{v}-\varepsilon}^{\bar{v}+\varepsilon} G\left(v\right) f^{\gamma B+1:B}\left(v\right) dv + o(1/B).$$

Making the change of variables $y = F(v)$ and again using $\hat{G}$ we obtain

$$V_G(\gamma B, B) = \int_{F(\bar{v}-\varepsilon)}^{F(\bar{v}+\varepsilon)} \hat{G}\left(F^{-1}\left(y\right)\right) r^{\gamma B:B-1}(y) dy + o(1/B).$$

Since the function $H\left(y\right) = \hat{G}\left(F^{-1}\left(y\right)\right)$ is four times differentiable we can use a Taylor expansion in a neighborhood of $\bar{y} = 1-\gamma$ and obtain

$$V_G\left(\gamma B, B\right) = \int_{F(\bar{v}-\varepsilon)}^{F(\bar{v}+\varepsilon)} \left(\sum_{k=0}^{3} \frac{1}{k!} H^{(k)}(\bar{y})(y-\bar{y})^k + \frac{1}{24} H^{(4)}(\varsigma(y))(y-\bar{y})^4 \right) r^{\gamma B:B-1}(y) dy + o\left(\frac{1}{B}\right)$$

for $\varsigma\left(y\right) \in \left[F\left(\bar{v}-\varepsilon\right), F\left(\bar{v}+\varepsilon\right)\right]$. The assumption that *f* is three times differentiable and bounded away from zero on the interior of its support is sufficient to ensure that there exists a constant *N* such that $H^{(4)}(\varsigma(y)) < N$ for all $y \in \left[F\left(\bar{v}-\varepsilon\right), F\left(\bar{v}+\varepsilon\right)\right]$.

The integral on the right-hand side is a sum of five sub-integrals I_1, I_2, I_3, I_4, and I_5 that can be easily evaluated or bounded because they are essentially moments of the $\gamma B : B-1$ order statistic of the uniform distribution. An argument almost identical to that used to show that J_2 is $o(1/B)$ can be used to show that the error from evaluating the integrals over $\left[0,1\right]$ instead of $\left[F\left(\bar{v}-\varepsilon\right), F\left(\bar{v}+\varepsilon\right)\right]$ is of order $o(1/B)$.[21]

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[21] In the case of I_5, we change the limits of integration only after substituting an upper bound for $H^{(4)}(\varsigma(y))$.

To evaluate $I_1 - I_4$ we expand the integrals as

$$\int_0^1 \frac{1}{k!} H^{(k)}(\bar{y})(y-\bar{y})^k r^{\gamma B:B-1}(y)dy = \frac{1}{k!} H^{(k)}(\bar{y}) \sum_{n=0}^{k} \left(\binom{k}{n} (-\bar{y})^{n-k} \int_0^1 y^k r^{\gamma B:B-1}(y)dy \right),$$

and note that the k^{th} moment of the $\gamma B : B-1^{\text{st}}$ order statistic of a uniform distribution is $\int_0^1 y^k r^{\gamma B:B-1}(y)dy = \frac{(B-\gamma B)(B-\gamma B+1)\cdots(B-\gamma B+k-1)}{B(B+1)\cdots(B+k-1)}$. Making this substitution gives

$$\frac{I_1}{H(\bar{y})} = \int_0^1 r^{\gamma B+1:B}(y)dy + o\left(\frac{1}{B}\right) = 1 + o\left(\frac{1}{B}\right).$$

$$\frac{I_2}{H'(\bar{y})} = \int_0^1 y r^{\gamma B+1:B}(y)dy - \bar{y} + o\left(\frac{1}{B}\right) = \frac{B-\gamma B}{B} - (1-\gamma) + o\left(\frac{1}{B}\right) = o\left(\frac{1}{B}\right)$$

$$\frac{I_3}{\frac{1}{2}H''(\bar{y})} = \frac{(B-\gamma B)(B-\gamma B+1)}{B(B+1)} - 2\frac{(B-\gamma B)}{B}(1-\gamma) + (1-\gamma)^2 + o\left(\frac{1}{B}\right) = \frac{\gamma(1-\gamma)}{B} + o\left(\frac{1}{B}\right)$$

$$\frac{I_4}{\frac{1}{6}H^{(3)}(\bar{y})} \le \frac{(B-\gamma B)(B-\gamma B+1)(B-\gamma B+2)}{B(B+1)(B+2)} - 3\frac{(B-\gamma B)(B-\gamma B+1)}{B(B+1)}(1-\gamma)$$
$$+3\frac{(B-\gamma B)}{B}(1-\gamma)^2 - (1-\gamma)^3 + o\left(\frac{1}{B}\right)$$
$$= \left((1-\gamma)^3 \left(1-\frac{3}{B}\right) + \frac{3}{B}(1-\gamma)^2 \right) - 3\left((1-\gamma)^3 \left(1-\frac{1}{B}\right) + \frac{1}{B}(1-\gamma)^2 \right) + 3(1-\gamma)^3 - (1-\gamma)^3 + o\left(\frac{1}{B}\right) = o\left(\frac{1}{B}\right).$$

Finally,

$$\frac{|I_5|}{\frac{1}{24}N} \le \frac{(B-\gamma B)(B-\gamma B+1)(B-\gamma B+2)(B-\gamma B+3)}{B(B+1)(B+2)(B+3)} - 4\frac{(B-\gamma B)(B-\gamma B+1)(B-\gamma B+2)}{B(B+1)(B+2)}(1-\gamma)$$
$$+6\frac{(B-\gamma B)(B-\gamma B+1)}{B(B+1)}(1-\gamma)^2 - 4\frac{(B-\gamma B)}{B}(1-\gamma)^3 + (1-\gamma)^4 + o\left(\frac{1}{B}\right).$$

After some algebra, one can show that this is $o(1/B)$.

Combining all terms we obtain:

$$V_G(\gamma B, B) = \frac{1}{\gamma} H(1-\gamma) + \frac{H''(1-\gamma)}{2} \frac{1-\gamma}{B} + o\left(\frac{1}{B}\right) = \hat{G}(\bar{v}) + \frac{H''(1-\gamma)}{2} \frac{(1-\gamma)}{B} + o\left(\frac{1}{B}\right)$$

The second derivative of $H(y) = (1-y)\hat{G}\left(F^{-1}(y)\right)$ evaluated at $\bar{y} = 1-\gamma$ can be calculated as:

$$H''(1-\gamma) = \frac{\hat{G}''(\bar{v})}{f(\bar{v})^2} - \hat{G}'(\bar{v}) \frac{f'(\bar{v})}{f(\bar{v})^3}$$

By plugging in the corresponding $\hat{G}$ function we can calculate seller and total surplus, and buyer utility. A little bit of algebra yields the expressions in the statement of the theorem.

We finally extend the integer-valued utility functions of buyers and sellers to $\mathbb{R}^2$. Consider any point $(S, B) \in \mathbb{R}^2$. This point lies inside the rectangle described by the vectors $v_1 = (\lfloor S \rfloor, \lfloor B \rfloor)$, $v_2 = (\lfloor S \rfloor, \lfloor B \rfloor + 1)$, $v_3 = (\lfloor S \rfloor + 1, \lfloor B \rfloor)$ and $v_4 = (\lfloor S \rfloor + 1, \lfloor B \rfloor + 1)$. There exists a unique convex decomposition of $(S, B) = \sum_{i=1}^{4} \lambda_i v_i$ such that $\lambda_i \geq 0$ and $\sum_{i=1}^{4} \lambda_i = 1$. We can then define the continuous extensions $u_b(S, B) = \sum_{i=1}^{4} \lambda_i u_b(v_i)$ and $u_s(S, B) = \sum_{i=1}^{4} \lambda_i u_b(v_i)$ of buyers' and sellers' utility functions. The above convergence proof gives us also uniform convergence to the approximation over $\Gamma = [\underline{\gamma}, \bar{\gamma}]$. QED

Proof of Proposition 12: The u_b and u_s functions clearly satisfy assumption (A1); Ellison and Fudenberg (2002) shows that (A2) and (A3) are satisfied when the economy is sufficiently large. It therefore suffices to show that for *B* sufficiently large, all intersections with $B_1 < B/2$ of the curves $S_1^{B1}(B_1)$ and $S_1^{S_1}(B_1)$ for which (B1) and (S1) hold with equality have

B_1/B within ε of $(1-\gamma)/4$. (B1) holds with equality if and only if $\frac{S_1}{B_1} = \frac{S_2}{B_2+1}$. Hence,

$S_1^{B1}(B_1) = \frac{B_1}{B+1}\gamma B$.

The set of B_1 with $S_1^{B1}(B_1) = S_1^{S1}(B_1)$ is thus the set of solutions to

$$u_s(\frac{B_1}{B+1}\gamma B, B_1) - u_s(\gamma B - \frac{B_1}{B+1}\gamma B + 1, B - B_1) = 0.$$

Suppose that the LHS of this equation can be approximated by a function $f(B_1/B)$ in the sense that for some $r(z) > 0$

$$\lim_{B\to\infty} \sup_{|\tilde{z}-z|<r(z)} B\left[f(\tilde{z}) - u_s(\frac{\tilde{z}B}{B+1}\gamma B, \tilde{z}B) + u_s(\gamma B - \frac{\tilde{z}B}{B+1}\gamma B + 1, B - \tilde{z}B) \right] = 0$$

for all $z \in (0,1/2]$, i.e., the function $f(z)$ converges to the LHS of the above equation locally uniformly. If $\{\underline{B}_1(B)\}$ is a sequence of solutions to $S_1^{B1}(\underline{B}_1(B)) = S_1^{S1}(\underline{B}_1(B))$ for B=1, 2, … and $\underline{z}$ is a subsequential limit point of $\underline{B}_1(B)/B$ then $f(\underline{z}) = 0$. Hence, it suffices to show that for some choice of the approximating function f the unique solution to $f(z) = 0$ in $(0,1/2]$ is $(1-\gamma)/4$.

We approximate the equation for (S1) holding with equality using the formula noted earlier: $u_s(S_1, B_1) - u_s(S_2+1, B_2) = \ln\left(\frac{B_1}{S_1}\right) - \ln\left(\frac{B_2}{S_2+1}\right) - \frac{1}{2}\left(\frac{1}{S_1} - \frac{1}{B_1}\right) + \frac{1}{2}\left(\frac{1}{S_2+1} - \frac{1}{B_2}\right) + o\left(\frac{1}{B}\right)$.

For $S_1 = \frac{B_1}{B+1}\gamma B$ we have

$$\ln\left(\frac{B_1}{S_1}\right) - \ln\left(\frac{B_2}{S_2+1}\right) = \ln\left(\frac{B+1}{\gamma B}\right) - \ln\left(\frac{B_2}{B_2+1}\frac{B+1}{\gamma B}\frac{S_2}{S_2+1}\right) = \frac{1}{B_2} + \frac{1}{S_2} + o\left(\frac{1}{B}\right).$$ Plugging

in zB for B_1, $(1-z)B$ for B_2, $\gamma z B^2/(B+1)$ for S_1, etc. and approximating to first order, e.g.,

$B(1/B_1)=1/z$, $B(1/S_1)=B(B+1)/(\gamma zB^2)=1/\gamma z+o(1)$, etc., gives

$$2B(u_s(\frac{zB}{B+1}\gamma B,zB)-u_s(\gamma B-\frac{zB}{B+1}\gamma B,B-zB))=\frac{2}{(1-z)}+\frac{2}{\gamma(1-z)}-\frac{1}{\gamma z}+\frac{1}{z}+\frac{1}{\gamma(1-z)}-\frac{1}{(1-z)}+o(1).$$

Set $Bf(z)$ equal to the function on the RHS without the error term. It can be checked that convergence of $f(z)$ to the expression on the LHS is indeed locally uniform because the error term will not blow up as z is varied a little bit. Multiplying $f(z)$ through by $\gamma z(1-z)$ we find $f(z)=0$ if and only if $2\gamma z+2z-(1-z)+\gamma(1-z)+z-\gamma z=0$,

which reduces to $z=(1-\gamma)/4.$ QED

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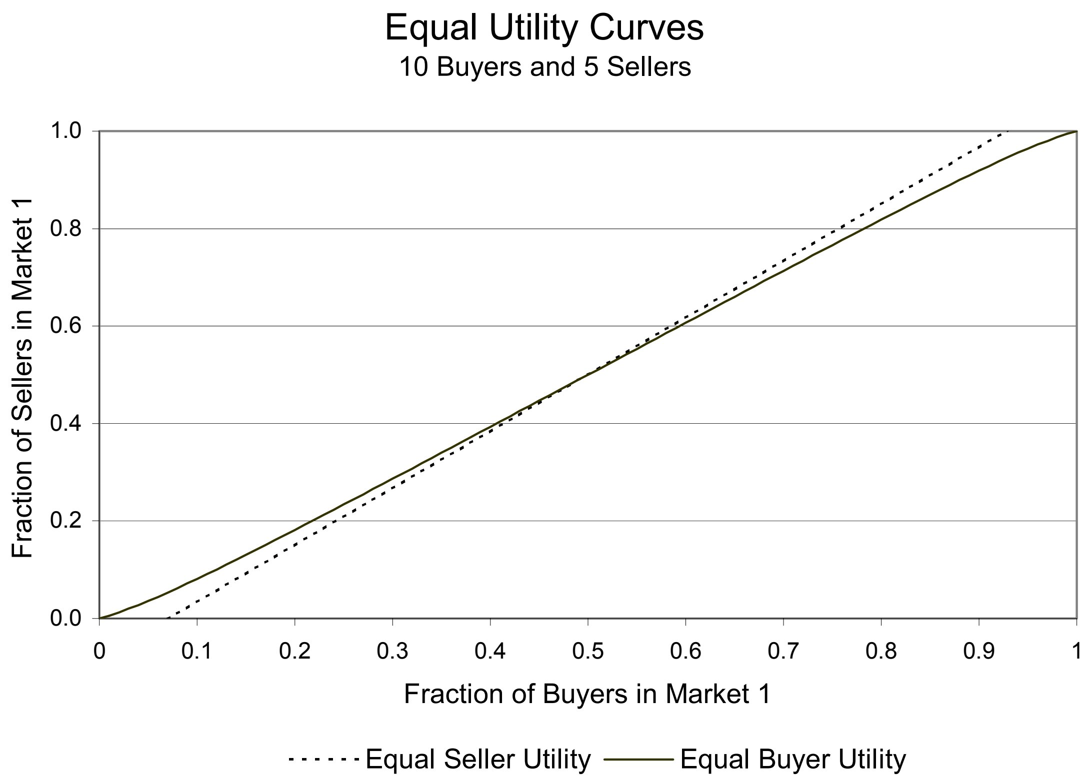


Equal Utility Curves
10 Buyers and 5 Sellers
Fraction of Sellers in Market 1
1.0
0.8
0.6
0.4
0.2
0.0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Fraction of Buyers in Market 1
Equal Seller Utility
Equal Buyer Utility

Figure 1

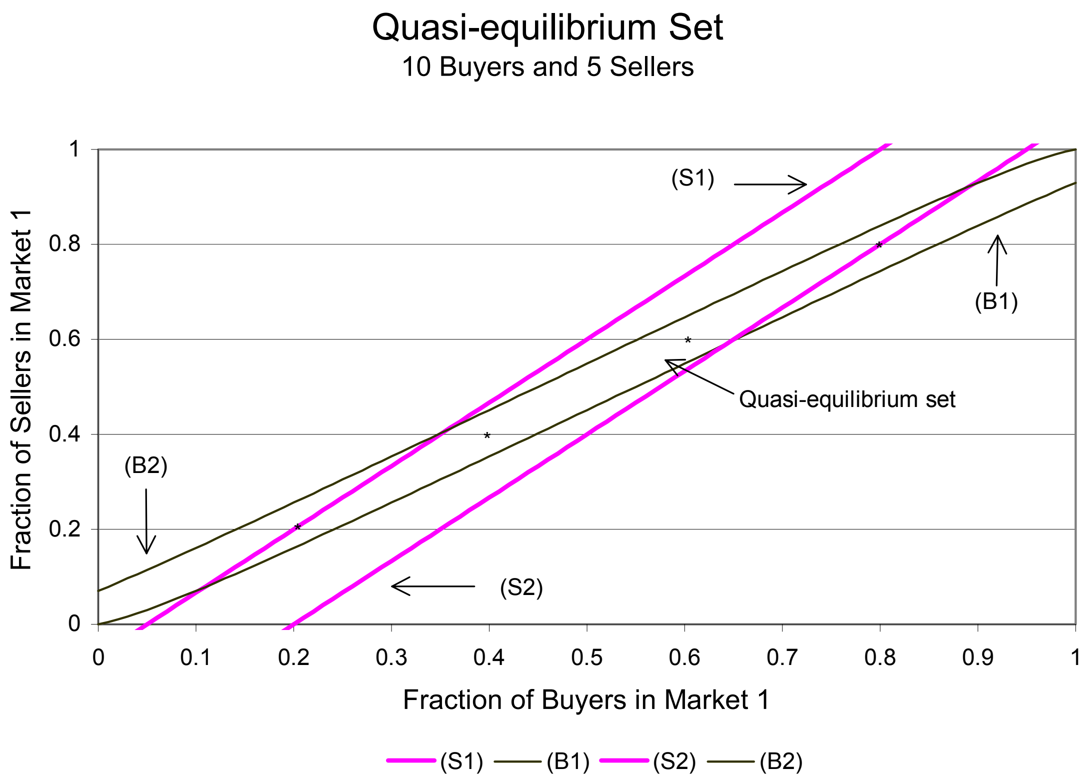
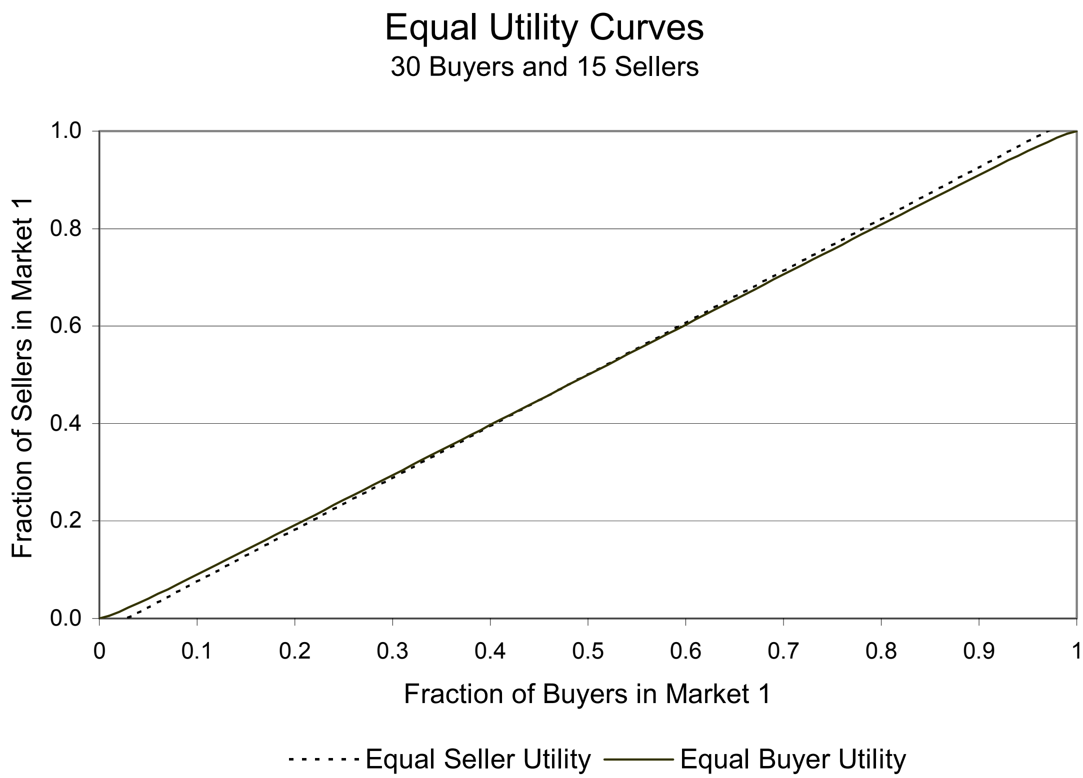

Quasi-equilibrium Set
10 Buyers and 5 Sellers
Fraction of Sellers in Market 1
Fraction of Buyers in Market 1
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
(S1)
(B1)
(B2)
(S2)
Quasi-equilibrium set
(S1) (B1) (S2) (B2)

Figure 2



Equal Utility Curves
30 Buyers and 15 Sellers
1.0
0.8
0.6
0.4
0.2
0.0
Fraction of Sellers in Market 1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Fraction of Buyers in Market 1
Equal Seller Utility
Equal Buyer Utility

Figure 3

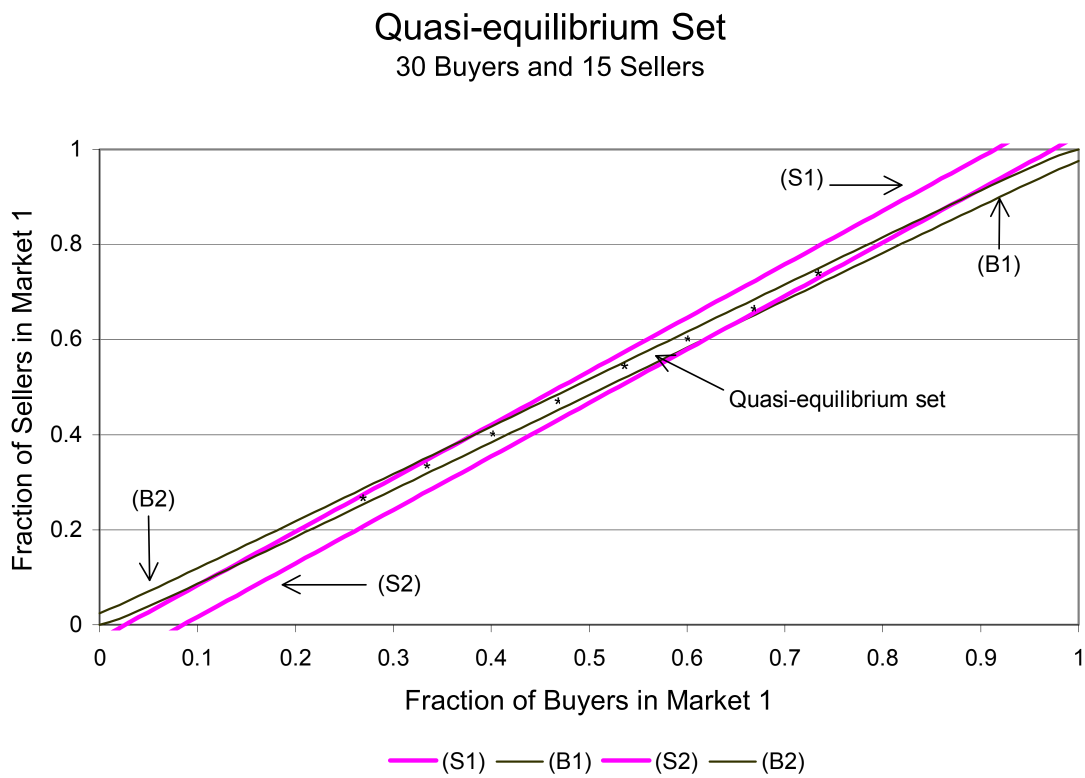

Quasi-equilibrium Set
30 Buyers and 15 Sellers
Fraction of Sellers in Market 1
Fraction of Buyers in Market 1
0
0.2
0.4
0.6
0.8
1
0.1
0.3
0.5
0.7
0.9
(S1)
(B1)
(S2)
(B2)
Quasi-equilibrium set

Figure 4