Fair Allocation Rules

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1 Introduction

Economists are often criticized for paying little attention to issues of fairness in distribution. It is indeed true that many shy away from making recommendations on the fair allocation of goods and services, perhaps fearing the necessary reliance on value judgements of such recommendations. However, in the last thirty years, a variety of formal criteria of economic justice have been introduced that have broad conceptual appeal as well as significant operational power. Our purpose is to review this literature. Mathematically, the objects of study are mappings that associate with each economy in some domain a non-empty subset of its set of feasible allocations. These mappings are called "solutions" or "rules".

The Bergson-Samuelson social welfare functions (Bergson, 1938; Samuelson, 1938), a central tool in traditional welfare economics, do provide the basis for answers to distributional questions in the form of complete orderings defined on the space of vectors of utilities (Chapter 14). However, they are in general not ordinal, and in fact they often rely on interpersonal comparisons of utility. The conceptual and practical issues associated with the measurement of an agent's utility, the choice of scales in which to make interpersonal comparisons of utility comparisons, and the manner in which to make them, are discussed in Chapter 14.

The social choice literature that originated in Arrow's work (1951) (Part I), partly as a criticism of the Bergson-Samuelson approach, does not require utility information, but one may argue that its formulation is too abstract to be directly relevant to the understanding of concrete resource allocation problems such as problems of fair allocation. Indeed, it ignores information that gives such problems their specificity. The mathematical structure of consumption spaces is not retained, so that restrictions on preferences that would be natural to impose cannot be formulated. Rules cannot be defined that take into account that information, nor can requirements on rules that are meaningful for their evaluation in each particular context, be expressed.

It is only in 1967 that an ordinal equity criterion designed for the evaluation of choices in concretely specified allocation models was first proposed, "no-envy" (Foley, 1967). This criterion has since been the object of a large number of studies. Its limitations have been uncovered and alternative notions offered, in part to remedy these limitations. Although it remains important, we will see that the literature has developed much beyond it.

On what basis should we evaluate a solution? First of all, of course, is our intuition about what is "fair". We should also understand when it is compatible with other fundamental requirements, such as efficiency. If certain minimal "rights" or "guarantees" have to be offered to agents, does it protect these rights and does it offer these guarantees? Another consideration is the scope of its applicability. We should ask whether it performs well on "classical" domains of privately appropriable goods, that is, when these goods are infinitely divisible and preferences satisfy standard assumptions of continuity, monotonicity, and convexity, and production sets (when included in the specification of economies) are similarly well-behaved. But we should also inquire whether it can accommodate wider classes of situations, when public goods or indivisible goods are present, when preferences may be satiated or exhibit consumption externalities, and when technologies are subject to increasing-returns-to-scale or externalities.

How adaptable is the solution to the evaluation of trades, as opposed to that of allocations, and to the evaluation of how fairly groups, as opposed to individuals, are treated? Its informational demands and ease of computation are also relevant issues. Is it appropriately responsive to changes in the parameters defining the economy, such as increases in the resources available, improvements in technology, variations in the population of agents involved, or changes in their preferences? Does it give agents the incentive to exert themselves or to provide accurate information about their tastes or the technologies they are familiar with? How vulnerable is it to manipulation? And can it be implemented by well-chosen game forms? All of these are questions that should be addressed.

This survey deals only with concretely specified economic models and ordinal solutions. Yet, it is undeniable that distributional judgments are often based on utility information, and even involve interpersonal comparisons of utility. And great is the intellectual appeal of the all-encompassing theories of the kind envisioned by Arrow (1951), Sen (1970), Rawls (1971), and others. However, by targeting more narrowly resource allocation problems, a lot more can be said, and this, without invoking concepts of utility. We will attempt here to find out how far such an approach can take us. The debate about which variables one should focus on when discussing fairness (resources, welfares, utilities, opportunities, functionings) has involved many writers (Dworkin, 1981a,b; Cohen, 1989, 1990; Sen, 1970; Roemer, 1996).

Here, we center our attention on resources and opportunities, understood in their physical sense.

The scope of our study is limited in another way, to models in which agents' preferences play a central role. Although it is methodologically relevant, we ignore the literature concerning certain models of cost allocation in which demands are taken as fixed data, as they have been the object of other surveys in this volume, and the literature on the adjudication of conflicting claims (Moulin, Chapter 6; Thomson, 2003). Also, we only make brief reference to models whose mathematical structures almost by definition preclude the existence of allocations meeting any reasonable definition of fairness. An example is when indivisible goods have to be allocated but monetary compensations are not possible. Finally, and although we see their study as an integral part of the program under discussion here, we leave strategic issues aside. They too are the subject of other chapters (Barberà, Chapter 23).

Earlier surveys covering some of the same ground as the present one are Thomson and Varian (1985), Young (1985), Arnsperger (1994), and Moulin and Thomson (1997). Book-length treatments are Kolm (1972, 1997), Young (1994), Moulin (1995, 2003), Thomson (1995c), Brams and Taylor (1996), Fleurbaey (1996b), and Roemer (1996).

As far as the search for meaningful social orderings in the context of resource allocation is concerned, we note that an important axiomatic literature has recently been developed (Fleurbaey and Maniquet and coauthors, Chapter 21). These advances are conceptually related to the program we describe here. A precursor of this work is Goldman and Sussangkarn (1978) and recent contributions are by Tadenuma (2001, 2005).

2 A note on the approach followed

We use the classical model to introduce the central concepts. In subsequent sections, we revisit these concepts in the context of several other models. Much of the literature we survey is axiomatic. For a discussion of this approach in this context we refer to Thomson (2001), but a short presentation may be useful at this point.

Our goal is to identify well-behaved solutions and rules. Requirements on these mappings, given the mathematical expressions of **axioms**, are formulated. Their logical relations are clarified and their implications, when imposed singly and in various combinations, are explored. For each such combination, do solutions exist that satisfy all of them? If yes, can one characterize the class of solutions satisfying them?

Requirements can be organized into two main categories. “Punctual” requirements apply to each economy separately; “relational” requirements relate choices made across economies. The first category can be subdivided: one subcategory consists of bounds on welfares defined agent-by-agent, in an intra-personal way; some are lower bounds, offering agents welfare guarantees; others are upper bounds, specifying ceilings on their welfares. The other subcategory consists of concepts based on inter-personal comparisons of bundles, or more abstractly, “opportunities” or “circumstances”, involving exchanges of, or other operations performed on, these objects.

The category of relational requirements can also be subdivided. First are various expressions of the central idea of solidarity: when the environment in which agents find themselves changes, and if no one in particular is responsible for the change, (or no one in a particular group of agents is responsible for it,) the requirement is that the welfares of all agents, (or all agents in this particular group), should be affected in the same direction: these agents—we call them the “relevant” agents—should all end up at least as well off as they were initially, or they should all end up at most as well off. In implementing this general idea, the focus of each study is usually on one specific parameter entering the description of the environment under consideration. When this parameter belongs to a space equipped with an order structure, which is often the case, one can speak of the parameter being given a “greater” or “smaller” value in that order. Examples are the resources available (a point in a vector space), the technology (a subset of a vector space), and the population of agents present (a natural number). Then, depending on which assumptions are made on preferences, together with efficiency, the solidarity idea often implies a specific direction in which welfares should be affected. It tells us that all relevant agents should end up at least as well off as they were initially, or it tells us that they should all end up at most as well off. Thus, solidarity takes the form of what are usually called “monotonicity” requirements.

The other subcategory of relational requirements are expressions of the idea of robustness. They are motivated less by normative considerations than by the desire to prevent the solution from being too dependent on certain data of the problem. In that family are several notions of conditional invariance under changes in preferences, resources, technology, or populations. A number of those are particularly relevant to the understanding of strategic issues, but, as already mentioned, other contributors to this volume have reviewed the strategic component of the axiomatic literature. For that reason, we only make passing mention of studies involving both normative and strategic principles but in which the latter play a central role.

Although the general principles just discussed are few, for each particular model, some adaptation is often necessary. This is what makes the work interesting and challenging. Also, because models vary in their mathematical structures, the implications of a given principle may differ significantly from one to the other. For instance, monotonicity requirements are very restrictive for the classical model when imposed in conjunction with no-envy and efficiency, but not so in the context of allocation with single-peaked preferences. This survey should help readers gain an appreciation of this twin fact, the great generality of the principles invoked and the specificity of their implications for each model.

Important remark on notation and language. We examine in succession several classes of economies, starting with the canonical model which concerns the allocation of a bundle of unproduced goods among agents equipped with standard preferences. We introduce each issue in the context of this model, and study it in the context of several other models in subsequent sections. We consider economies with public goods, economies with indivisible goods, economies with single-peaked preferences, and economies in which the dividend is a non-homogenous continuum. When we turn to these models, and in order to save space, we do not rewrite all the definitions, unless the necessary adjustments are not straightforward. We hope that no confusion will result from taking this shortcut.

Our generic notation for an agent set is N and when this set is fixed, $N \equiv \{1, \ldots, n\}$. Some of our relational axioms involve variations of populations. To allow for such variations, we imagine then that agents are drawn from an infinite universe of "potential" agents, the set $\mathbb{N}$ of natural numbers. We denote by $\mathcal{N}$ the class of finite and non-empty subsets of $\mathbb{N}$, and N is now the generic element of $\mathcal{N}$. In the statements of many results, solutions and rules are required to satisfy certain axioms none of which involves changes in populations. Other relational axioms may be invoked however, specifying how rules should respond to changes in other parameters. In each case, the domain should be understood to be rich enough that changes in these parameters can actually take place.

Our generic notation for solutions is the letter φ. We place statements pertaining to strategic questions within square brackets to help relate our survey to the work reviewed in other chapters. We always assume preferences to be continuous (when continuity is meaningful), and omit this assumption from the formal statements of the theorems.

A solution is **single-valued**—we refer to it as a **rule** then—if for each economy in its domain, it selects a single allocation, and **essentially single-valued** if whenever it selects several allocations, these allocations are **Pareto-indifferent**, that is, all agents are indifferent between them. An *essentially single-valued* subsolution of a solution φ is a **selection from φ**.

3 The classical problem of fair division

Here is our first model. There are ℓ privately appropriable and infinitely divisible goods and a set $N \equiv \{1, \ldots, n\}$ of agents. Each agent $i \in N$ is described by means of a (continuous, as indicated previously) **preference relation** R_i defined on $\mathbb{R}^\ell_+$. The strict preference relation associated with R_i is denoted by P_i and the corresponding indifference relation by I_i. Let $R \equiv (R_i)_{i \in N}$ be the profile of preference relations.

Let $\mathcal{R}$ be our generic notation for a **domain of admissible preferences**. A preference relation is **classical** if it is (i) continuous: lower and upper contour sets are closed, (ii) monotonic: if $z'_i > z_i$, then $z'_i \; P_i \; z_i$,[1] and (iii) convex: upper contour sets are convex. On occasions, we assume preferences to be strictly monotonic: if $z'_i \geq z_i$, then $z'_i \; P_i \; z_i$.

Since preferences are continuous, we can represent them by continuous real-valued functions, and it is sometimes convenient to do so. For each $i \in N$, let $u_i: \mathbb{R}^\ell_+ \to \mathbb{R}$ be such a representation of agent i's preferences, and let $u \equiv (u_i)_{i \in N}$. Except in a few places in this exposition, these representations have no cardinal significance. Indeed, the theory developed here is based only on preferences.

The vector of resources available for distribution, the **social endowment**, is denoted $\Omega \in \mathbb{R}^\ell_+$.

Altogether, an **economy** is a pair $(R, \Omega) \equiv ((R_i)_{i \in N}, \Omega) \in \mathcal{R}^N \times \mathbb{R}^\ell_+$. Let $\mathcal{E}^N$ be our generic notation for a domain of economies.

[1] Vector inequalities: $x \geqq y$ means that for each $i \in N$, $x_i \geq y_i$; $x \geq y$ means that $x \geqq y$ and $x \neq y$; $x > y$ means that for each $i \in N$, $x_i > y_i$.

In an economy, resources are owned collectively, whereas in an **economy with individual endowments**, each agent starts out with a particular share of society's resources. Agent i's **endowment**, a vector $\omega_i \in \mathbb{R}^{\ell}_{+}$, is usually interpreted as a bundle that he has the right to dispose of as he wishes. However, this interpretation, which strongly suggests that he is entitled to a welfare that is at least the welfare he experiences when consuming his endowment, is not the only possible one.[2] Formally, $\omega \equiv (\omega_i)_{i \in N}$ is simply a reference allocation on which the final choice is allowed, or required, to depend. A generic economy is now a pair $e \equiv (R, \omega)$ with $R \in \mathcal{R}^N$ and $\omega \in \mathbb{R}^{\ell N}_{+}$.

We distinguish between the problem of fairly allocating a social endowment from the problem of fairly **re**-allocating individual endowments. A **feasible allocation for (R, Ω)** is a list $z \equiv (z_i)_{i \in N} \in \mathbb{R}^{\ell N}_{+}$ such that $\sum z_i = \Omega$,[3] and a **feasible allocation for (R, ω)** is a list $z \equiv (z_i)_{i \in N} \in \mathbb{R}^{\ell N}_{+}$ such that $\sum z_i = \sum \omega_i$. The i-th component of a feasible allocation is agent i's **consumption bundle**. The equality sign appearing in the feasibility constraint indicates the absence of free disposal. When preferences are monotonic and efficiency is required, as is mostly the case here, this assumption entails no loss of generality. In Section 11, where we drop monotonicity, the equality takes real significance. Let $Z(e)$ be the set of feasible allocations of $e \in \mathcal{E}^N$. A feasible allocation of particular interest is **equal division**, $(\frac{\Omega}{|N|}, \ldots, \frac{\Omega}{|N|})$. We designate it as **ed**.

We recall that an allocation is (Pareto)-**efficient** if it is feasible and there is no other feasible allocation that **Pareto-dominates** it, which means that each agent finds it at least as desirable and at least one agent prefers. It is **weakly efficient** if there is no other feasible allocation that each agent prefers. The **Pareto solution** associates with each economy its set of efficient allocations.

Sometimes, we find it notationally convenient to assume that preferences are defined over the cross-product of the consumption spaces. Instead of statements of the form "$z_i \; R_i \; z'_i$," we write "$z \; R_i \; z'$." Also, given $z, z' \in Z(e)$, if for each $i \in N$, $z_i \; R_i \; z'_i$, we write $z \; R \; z'$, the statements $z \; P \; z'$ and $z \; I \; z'$ being understood in a similar way. When externalities in consumption are present, such notation is of course necessary.

[2] This is why we prefer avoiding the phrase "private ownership economy", which is commonly used.

[3] Unless otherwise indicated, a sum without explicit bounds should be understood to be carried out over all agents.

Quasi-linear preference profiles are of particular interest: for such a profile, there is a good whose consumption is usually unrestricted in sign (but sometimes non-negativity is imposed)—we always let it be good 1—such that all preferences can be given representations that are separably additive in good 1 on the one hand, and a function $v_i\colon \mathbb{R}_+^{\ell-1} \to \mathbb{R}$ of the remaining goods on the other. Designating by $x_i \in \mathbb{R}$ agent i's consumption of good 1 and by $y_i \in \mathbb{R}_+^{\ell-1}$ his consumption vector of the other goods, we can write $u_i(x_i, y_i) = x_i + v_i(y_i)$. Quasi-linear economies lend themselves to the application of the solution concepts developed in the rich theory of cooperative games with "transferable utility". Quasi-linearity applies to domains other than the classical domain by appropriately choosing the arguments of the v_i functions.

4 Equitable allocations

The simplest problem of fair division is when there is only one good and preferences are strictly monotonic. Since in this situation, efficiency is automatically satisfied, how we choose to perform the division is a reflection of our position on normative issues only. Indeed, preferences are identical then. Our choice is equal division. This is because, as already announced, we have decided to ignore intensities of satisfaction—we have no "utility" information. When agents differ in the effort they provide, their productive talents, and so on, adjustments may also have to be made, but for the time being, we ignore these complications. An important exception is Section 5.

4.1 Comparisons to equal division

If there is more than one good, equal division conflicts with efficiency, so other criteria have to be formulated. Our next proposals are based on comparisons to equal division.[4] The first comparison is expressed in physical terms. **No-domination of, or by, equal division**, says that no agent should receive a bundle that contains at least as much as equal division of each good, and more than equal division of at least one good (Thomson, 1995c), or a bundle

[4]Pazner and Schmeidler (1976) give several reasons why equal division would emerge in the "original position", and they propose a way of giving operational meaning to the Rawlsian objective of making the worse-off agent as well-off as possible.

that contains at most as much as equal division of each good, and less than equal division of at least one good.

The next comparison involves preferences: each agent should find his bundle at least as desirable as equal division. It has been advocated by many authors (see for instance Kolm, 1973, and in particular, Pazner, 1977).

Definition Given $e \equiv (R, \Omega) \in \mathcal{E}^N$, the allocation $\boldsymbol{z} \in \boldsymbol{Z(e)}$ **meets the equal-division lower bound in** $\boldsymbol{e}$, written as $\boldsymbol{z} \in \underline{\boldsymbol{B}}_{\boldsymbol{ed}}\boldsymbol{(e)}$, if $z\, R\, (\frac{\Omega}{|N|}, \ldots, \frac{\Omega}{|N|})$.[5]

The existence of efficient allocations meeting the equal-division lower bound is a straightforward consequence of the compactness of the feasible set and continuity of preferences.

We add two elementary requirements on a solution involving comparisons to equal division. First, if equal division is efficient, it should be chosen. Second, if equal division is efficient, any allocation that is Pareto-indifferent to it—these allocations are the ones that meet the equal-division lower bound—should be chosen (Thomson, 1987c). This second requirement is stronger but it is satisfied by most of the solutions that we will encounter:

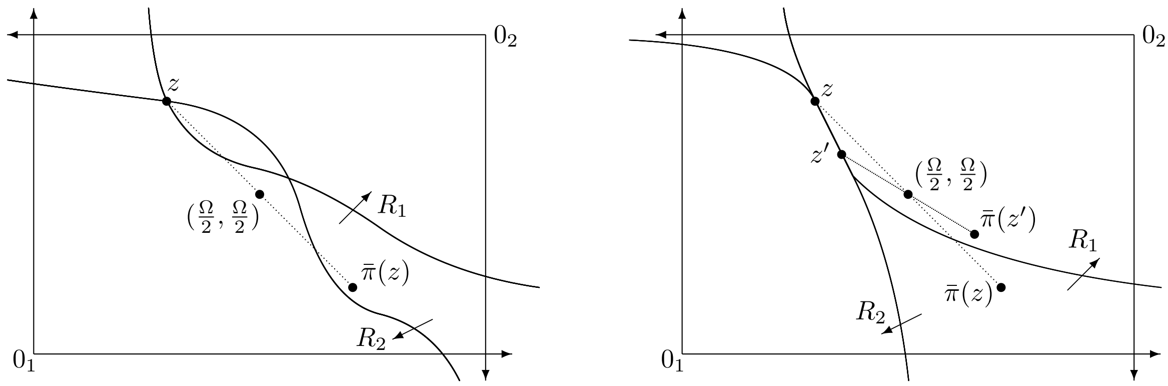
Property $\boldsymbol{\alpha}$: For each $e \equiv (R, \Omega) \in \mathcal{E}^N$, if $(\frac{\Omega}{|N|}, \ldots, \frac{\Omega}{|N|}) \in P(e)$, then $\varphi(e) \supseteq \underline{B}_{ed}(e)$.

4.2 No-domination

Another natural extension of our choice of equal division for the one-good case, this time involving inter-personal comparisons, but again formulated in physical terms, is that no agent should receive at least as much of all goods as, and more of at least one good than, some other agent. (Thomson, 1983a; Thomson and Varian, 1985. The notion appears as a formal solution in Moulin and Thomson, 1988):

Definition Given $e \equiv (R, \Omega) \in \mathcal{E}^N$, the allocation $\boldsymbol{z} \in \boldsymbol{Z(e)}$ **satisfies no-domination**, written as $\boldsymbol{z} \in \boldsymbol{D(e)}$, if there is no pair $\{i, j\} \subseteq N$ such that $z_i \geq z_j$.

[5]These allocations are often referred to as "individually rational from equal division", but we prefer not using this phrase since it implies that agents are entitled to equal division. This may not be a legitimate assumption.



(a) (b)

Figure 1: No-envy. In both panels, $N \equiv \{1, 2\}$. (a) A simple geometric test tells us that z is envy-free: each agent $i \in N$ finds his bundle z_i at least as desirable as its symmetric image with respect to the center of the Edgeworth box, $\bar{\pi}_i(z)$. (b) The no-envy and Pareto solution violates *Pareto-indifference*: here, z is envy-free and efficient and $z' \; I \; z$; yet, since agent 1 prefers z'_2 to z'_1, z' is not envy-free.

The no-domination requirement, which has the practical merit of being verifiable without knowing preferences, is a very weak one, and we will present more powerful ones. Most of them are based on comparisons of bundles involving preferences, and they differ from each other only in the specification of which comparisons are admissible. For no-envy, the admissible comparisons are between the allocation that is being evaluated and the allocations obtained by permuting its components. But we will define solutions based on comparisons that are restricted, for example to only efficient allocations, or extended, for example to certain infeasible allocations.

4.3 No-envy

Next, we require of an allocation that each agent should find his bundle at least as desirable as that of each other agent (Foley, 1967):[6]

Definition Given $e \equiv (R, \Omega) \in \mathcal{E}^N$, the allocation $\boldsymbol{z \in Z(e)}$ **is envy-free for** $\boldsymbol{e}$, written as $\boldsymbol{z \in F(e)}$, if for each pair $\{i, j\} \subseteq N$, $z_i \; R_i \; z_j$.

This definition can also be stated as follows. Let Π^N denote the class of permutations on N, with generic element π. Given $z \in Z(e)$, let $\pi(z)$ be the allocation whose i-th component is $z_{\pi(i)}$. Then, $z \in F(e)$ if and only if for

[6]The idea had been formulated by at least one previous writer. Tinbergen (1953) devotes a few pages to a discussion of the no-envy test, explaining that he had developed it in conversations with the Dutch physicist Ehrenfest. However, it is thanks to Foley that the criterion has become known, and this author is usually credited with it.

each $\pi \in \Pi^N$, $z \; R \; \pi(z)$ (in Figure 1a, the permutation exchanging agents 1 and 2 is denoted $\bar{\pi}$).

It is clear that if preferences are strictly monotonic, no-envy implies no-domination. Because equal division satisfies the definition, envy-free allocations always exist in the classical model.

Before we address the issue of existence of envy-free and efficient allocations, a comment on the terminology we have adopted may be useful. As usually understood, "envy" denotes a feeling that reflects negatively not only on the external circumstances in which people find themselves (here, their circumstances are defined by the bundles they receive), but also on preferences themselves. A number of writers have argued that reference to "envy" is justified only if externalities is consumption exist (Archibald and Donaldson, 1979). Translating into economic terminology the connotation that referring to an individual, say agent i, as being "envious" has in common language would require specifying his preferences as exhibiting consumption externalities that fail to satisfy the right monotonicity property with respect to agent j's consumption, where $j \neq i$. Chaudhuri (1985), following Nozick (1974), formalizes such a notion and relates it to other forms of externalities. He also establishes elementary properties of the binary "envy relation" that one can associate with each allocation, pointing out that it may not be transitive. Sussangkarn and Goldman (1980) suggest plausible specifications of these external effects, and establish various impossibilities in reconciling the standard approach, followed here, with an approach in which external effects are explicitly taken into account.

As we use the term, "envy" can occur even if externalities in consumption are not present. More neutral expressions, explicitly referring to the mathematical operation that is being performed in evaluating an allocation (such as "robustness under substitutions", "under transpositions", or "under permutations")[7] might be more accurate, but we chose to use the term that is common in the literature.[8]

To summarize the various possibilities that could arise when preferences do exhibit consumption externalities, and starting from a given allocation

[7]In Thomson (1983a) we use the phrase "permutation-acceptable" to designate a related concept. Gevers (quoted in Fleurbaey and Maniquet, 1996a) suggests the term "permutation-proof".

[8]In much of the early literature, the term "fair" was used to designate allocations that are envy-free and efficient (Schmeidler and Yaari, 1971). In common language, fairness has no efficiency connotation, and we will express the two requirements separately.

$z \in Z(e)$, we could check whether agent i is better off (i) after his bundle has been *switched* with agent j's bundle; (ii) after his bundle has been *replaced* by a bundle identical to agent j's; (iii) after agent j's bundle has been *replaced* by a bundle identical to agent i's; (iv) after an arbitrary *permutation* of the components of z.

Note that in (ii) and (iii), the list of bundles to which z is compared constitutes a feasible allocation only in the trivial case $z_i = z_j$. Also, a feature of "envy", as the term is commonly understood, is that it is *directed* against a specific individual. Thus, the operation described in (iv), where the i-th components of z and $\pi(z)$ may actually be the same, has nothing to do with envy, although it seems economically relevant: it reflects agent i's view on how the resources assigned to the other agents should be distributed between them.

Whether the feelings suggested in common language by the term "envy" should be acknowledged in the evaluation of an allocation has been the object of an interesting debate (Kolm, 1995, 1996; Fleurbaey, 1994).

From here on, we return to the standard case of preferences that do not exhibit consumption externalities. In such economies, do envy-free and efficient allocations exist? The answer is yes under classical assumptions.

Theorem 4.1 Domain: private goods; monotonic and convex preferences. *Envy-free and efficient allocations exist.*

The simplest way to prove Theorem 4.1 is to invoke the concept of an equal-division Walrasian allocation. These allocations are obviously envy-free:

Definition Given $e \equiv (R, \Omega) \in \mathcal{E}^N$, the allocation $\boldsymbol{z \in Z(e)}$ **is an equal-division Walrasian allocation for** $\boldsymbol{e}$, written as $\boldsymbol{z \in W_{ed}(e)}$, if there is $p \in \Delta^{\ell-1}$ such that for each $i \in N$, z_i is a maximizer of R_i in the budget set $B_i(p) \equiv \{z'_i \in \mathbb{R}^\ell_+ \colon pz'_i \leq p\frac{\Omega}{|N|}\}$.

Under the assumptions of Theorem 4.1, equal-division Walrasian allocations exist. In fact, any set of assumptions guaranteeing the existence of these allocations also guarantees that of envy-free and efficient allocations. Usually, convexity of preferences is included.[9] Although envy-free and efficient allocations may not exist if preferences are not convex (Varian, 1974),

[9] Actually, "constrained" equal-division Walrasian allocations (Hurwicz, Maskin, and Postlewaite, 1995), are envy-free as well, and under slightly stronger assumptions, efficient.

their existence can actually be derived from substantially weaker assumptions than those known to ensure the existence of Walrasian allocations. It should be noted however that, in addition to monotonicity of preferences, the main ones concern the structure of the efficient set, and not the primitives themselves. We state them in order of increasing generality: if an allocation z is efficient, (i) no other allocation is Pareto-indifferent to z (Varian, 1974);[10] (ii) the set of allocations that are Pareto-indifferent to z is convex (Svensson, 1983, 1994); (iii) the set of allocations that are Pareto-indifferent to z is contractible (Diamantaras, 1992).

At this point, we do not claim any particularly merit for the equal-division Walrasian solution except as a convenient means of delivering envy-free and efficient allocations. However, this solution stands out for its informational efficiency, as measured by the dimensionality of message spaces required for what is called in the theory of mechanism design, its "realization": in a formal sense that would require more machinery than is justified for this survey (see Hurwicz, 1977, and Mount and Reiter, 1974, for the theoretical foundations), the Walrasian mechanism has minimal dimension among all mechanisms realizing envy-free and efficient allocations. Moreover, it is the only such mechanism (Calsamiglia and Kirman, 1993). A number of other characterizations bringing out the informational merits of the Walrasian solution, such as the fact that it only depends on local information about preferences, or that it is invariant under certain "monotonic" transformations of preferences (Maskin, 1999; Gevers, 1986; Subsection 7.3) are available (Thomson, 1985, 1987c; Gevers, 1986; Nagahisa, 1991, 1992, 1994; Nagahisa and Suh, 1995; Yoshihara, 1998). Some of these characterizations involve no-envy but we will not elaborate, as the informational considerations on which they are based do not have a strong normative relevance. We only emphasize that by contrast, the no-envy criterion, as well as most of the other fairness notions surveyed here, cannot be checked on the basis of local information only. Marginal analysis, the fundamental tool of modern microeconomics, is often of little use when investigating fairness.

Beyond existence, we would like to understand the structure of the set of envy-free and efficient allocations. First, the no-envy and Pareto solution violates a requirement that is satisfied by virtually all solutions commonly

[10]Alternatively, one may assume of each preference relation that any non-zero bundle is preferred to the zero bundle, and on the preference profile that the Pareto set coincides with the weak Pareto set.

discussed. It says that if an allocation is chosen, then so should any allocation that all agents find indifferent to it:[11]

Pareto-indifference: For each $e \equiv (R, \Omega) \in \mathcal{E}^N$, and each pair $\{z, z'\} \subset Z(e)$, if $z \in \varphi(e)$ and $z' \; I \; z$, then $z' \in \varphi(e)$.

In the economy $e \equiv (R, \Omega)$ depicted in Figure 1b, $z \in F(e)$ (in fact, $z \in FP(e)$), and $z' \; I \; z$; yet, since $z'_2 \; P_1 \; z'_2$, $z' \notin F(e)$.[12]

Another basic requirement, obviously satisfied by the no-envy solution but by neither the equal-division lower bound solution nor by the no-domination solution, is that agents with the same preferences should be assigned bundles that are indifferent according to these preferences (not necessarily identical bundles, although, in the presence of efficiency, and if preferences are strictly convex, equality of bundles will result):[13]

Equal treatment of equals: For each $e \equiv (R, \Omega) \in \mathcal{E}^N$, each pair $\{i, j\} \subseteq N$, and each $z \in \varphi(e)$, if $R_i = R_j$, then $z_i \; I_i \; z_j$.

To obtain the entire set of envy-free allocations in an Edgeworth box, it suffices to identify for each agent the set of points each of which he finds indifferent to its symmetric image with respect to equal division. This set is (i) a strictly downward-sloping continuous curve (ii) that passes through equal division and is symmetric with respect to that point (Kolm, 1972; Baumol, 1982, 1986; Thomson, 1982; Kolpin, 1991a).[14] An allocation is envy-free if it is on or above each agent's envy boundary (Figure 2). In general, the envy-free zone is the disconnected union of closed sets, one of which contains the set of allocations meeting the equal-division lower bound.

Let us add efficiency. Under standard assumptions (if preferences are continuous, strictly monotonic, and strictly convex), the efficient set is a continuous curve connecting the two origins, and it intersects the envy-free set along a curvi-linear segment. This segment is contained in the component

[11] The usefulness of this condition in the context of the problem of fair division is pointed out by Thomson (1983a) and Gevers (1986). The condition has played an important role in recent literature.

[12] In fact, preferences could be drawn in such a way that z' violates no-domination.

[13] This condition is often referred to as "horizontal equity".

[14] Conversely, any curve with properties (i) and (ii) is the envy boundary of some monotonic and convex preference relation, for instance, one with right-angle indifference curves (Thomson, 1995c).

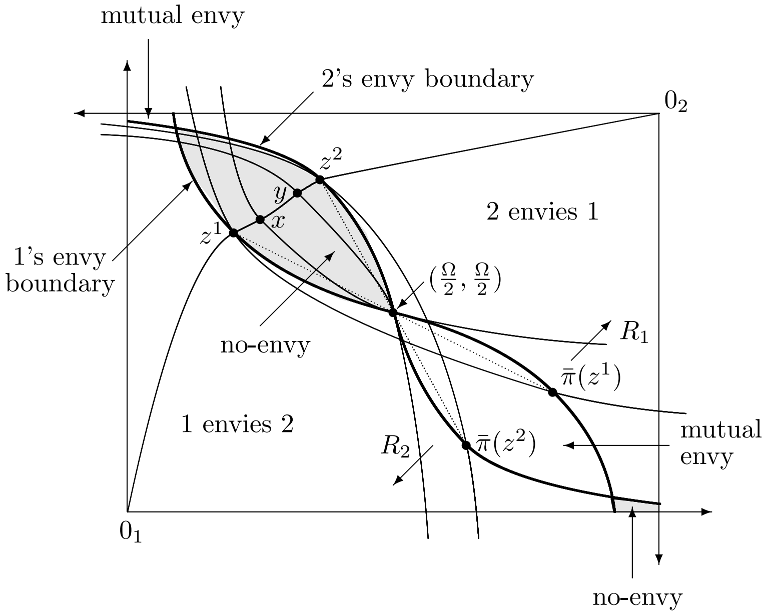


Figure 2: Determining the set of envy-free and efficient allocations in an Edgeworth box: it is the set of allocations that are above each agent's envy boundary. In the example, it is the union of two disconnected sets (the two shaded areas).

of the envy-free set to which equal division belongs. It often contains allocations that do not meet the equal-division lower bound. In Figure 2, these allocations constitute the curvi-linear segments from x to z^1 and from y to z^2.

The above remarks indicate that the envy-free set does not have a simple structure, even for $|N| = 2$. Unfortunately, if $|N| > 2$, things get worse. In particular, an allocation meeting the equal-division lower bound is not necessarily envy-free.

However, the main limitation of the no-envy concept is that in production economies, envy-free and efficient allocations may not exist, even under very standard assumptions on preferences and technologies. This difficulty occurs as soon as productivities are allowed to differ across agents (Section 5).

4.4 Concepts related to no-envy

Here, we discuss several variants and extensions of no-envy. First is **average no-envy**: each agent should find his bundle at least as desirable as the average of what the others receive. Indeed, one could argue that how the resources allocated in total to these agents have been distributed among them should be of no concern to him (Thomson, 1979, 1982; Baumol, 1982;

Kolpin, 1991b).

If $|N| = 2$, the criterion coincides with no-envy. Under convexity of preferences, it is less restrictive than the equal-division lower bound, but under certain regularity conditions, in a large economy (for example in a replica economy), it is "approximately" equivalent to it. An allocation may be envy-free and efficient and yet not average envy-free. Conversely, an allocation may be average envy-free and efficient and yet not envy-free.[15]

A stronger requirement, **strict no-envy**, involves subgroups: each agent should find his bundle at least as desirable as the average of the bundles received by any subgroup of the other agents (Zhou, 1992). A further strengthening is obtained by dropping the proviso that the agent not be a member of the comparison group.

The following requirement includes all of the above as special cases (Kolm, 1973).[16] Given $i \in N$, we specify (i) a subgroup $G_i \subseteq N$ of agents with whose bundles a comparison of agent i's bundle is judged relevant, and (ii) a set Λ_i of admissible vectors of weights $\lambda \in \Delta^{G_i}$ to be used in these comparisons.[17] Let $(\mathcal{G}, \Lambda) \equiv (G_1, \ldots, G_n, \Lambda_1, \ldots, \Lambda_n)$.

Definition (Kolm, 1973) Given $e \equiv (R, \Omega) \in \mathcal{E}^N$, the allocation $\boldsymbol{z} \in \boldsymbol{Z}(\boldsymbol{e})$ **is** $(\boldsymbol{\mathcal{G}}, \boldsymbol{\Lambda})$**-envy-free for** $\boldsymbol{e}$ if for each $i \in N$ and each $\lambda \in \Lambda_i$, $z_i \; R_i \; \sum_{j \in G_i} \lambda_j z_j$.

Suppose $G_i \equiv N \backslash \{i\}$. Then, if each Λ_i is the set of unit vectors in Δ^{G_i}, the resulting definition coincides with no-envy, and if Λ_i contains only the vector of equal coordinates, we obtain average no-envy. If for each $i \in N$, $G_i = N$ and $\Lambda_i = \Delta^N$, then the requirement is that each agent should find his bundle at least as desirable as any point in the convex hull of all assigned bundles (Kolm, 1996).[18] Strictly envy-free allocations are also obtained as a special case. Equal-division Walrasian allocations satisfy the definition no matter what $(\mathcal{G}, \Lambda)$ is (Kolm, 1973).

[15]For each agent, one can define a boundary above which an agent's consumption should lie for it to pass the average no-envy test. The properties of this boundary are established in Baumol (1986) and Thomson (1982). The boundary is analogous to the envy boundary encountered earlier but now the construction can be performed for any number of agents.

[16]Kolm uses the phrase "super envy-free".

[17]By the notation $\mathbb{R}^{G_i}$, we mean the cross-product of $|G_i|$ copies of $\mathbb{R}$, indexed by the members of G_i. Kolm actually allows the weights to be negative.

[18]Kolm uses the phrase "super equity".

Some solutions are based on comparing across agents the number of agents whom each agent envies and the number of agents who envy him. One such "counting requirement", **balanced envy**, is that for each agent, these two numbers should be equal (Daniel, 1975).[19] It is particularly appealing in production economies, where there may be no envy-free and efficient allocations (Section 5). Indeed, the existence of allocations with balanced envy in such economies can be established more generally than is standard for other concepts (as in the variants of Theorem 4.1 discussed after its statement). In economies without production, the main assumption for existence is not convexity of preferences, but that no two Pareto-efficient allocations be Pareto-indifferent. (A refinement is proposed by Fleurbaey, 1994, and proven to be non-empty).

The concept has been criticized because it is theoretically compatible with the existence of a large number of occurrences of envy (Pazner, 1977). However, in situations where envy cannot be avoided, it is natural to attempt to distribute it "uniformly" across agents. Other natural ideas are to require of an allocation that all agents should envy the same number of agents, or that all agents should be envied by the same number of agents. But neither will do, as soon as efficiency is imposed, a consequence of the following proposition:

Proposition 4.1 (Varian, 1974; Feldman and Kirman, 1974) Domain: feasible set is closed under permutations of the components of allocations. *At an efficient allocation, at least one agent envies no one, and at least one agent is envied by no one.*

Proposition 4.1 certainly reinforces the appeal of the balanced-envy criterion but there are other ways to deal with the non-existence of envy-free and efficient allocations in production economies (Section 5). Varian (1976) invokes the proposition to render operational the Rawlsian objective of making the worse-off agent as well off as possible, defining a worse-off agent as one whom no-one envies, and "making that agent as well off as possible" being interpreted as minimizing the number of agents whom he envies—no one if that is possible. (Then, an envy-free allocation is obtained.)

Other notions have been proposed in which each agent's preference relation is used to compare the bundles received by any two other agents. A suggestion is to require of an allocation that there should not be two agents

[19] Daniel uses the term "just".

such that all agents, if consuming the bundle assigned to the first one, would be worse off than if consuming the bundle assigned to the second one (van Parijs, 1990. Iturbe-Ormaetxe and Nieto, 1996a, and Fleurbaey, 1994, generalize the idea.)

4.5 Selections and rankings

As we have seen, in exchange economies, envy-free and efficient allocations exist very generally, and the set they constitute may even be quite large. In such situations, we would like to be able to identify which allocations are preferable in the envy-free set, that is, to refine the no-envy concept, perhaps to rank all allocations in terms of the extent to which they satisfy no-envy. However, strengthening it will aggravate the non-existence difficulty that we will encounter in production economies, and unfortunately, the concept does not seem to lead directly to a ranking of all feasible allocations in terms of equity. Of course, we could (i) declare socially indifferent all allocations at which no one envies anyone, (ii) declare socially indifferent all allocations at which someone envies someone else, and (iii) declare any allocation at which no one envies anyone socially preferred to any allocation at which someone envies someone else. But this ranking would have only two indifference classes and thus would provide a trivial answer to our question.[20] Finer rankings would be desirable.

One suggestion has been to measure aggregate envy at an allocation z by the number of pairs $\{i, j\} \subseteq N$ such that $z_j \; P_i \; z_i$ (Feldman and Kirman, 1974). Other proposals involve cardinal measurements of welfare differences: after choosing for each agent $i \in N$ a numerical representation of his preferences, u_i, the extent to which agent i is envious is quantified by the sum $e_i(z) \equiv \sum_j \max\{0, u_i(z_j) - u_i(z_i)\}$. (When utility information is available, one may argue that these are the functions that should be used for that purpose.) Aggregate envy is then evaluated by the expression $\sum e_i(z)$ (Feldman and Kirman, 1974). Alternatively, we may also want to take into account the extent to which agent i prefers his bundle to the bundles of the agents whom he does not envy. Then, defining $\bar{e}_i(z) \equiv \sum_{j \in N}[u_i(z_j) - u_i(z_i)]$, aggregate envy is evaluated by the expression $\sum \bar{e}_i(z)$. Finally, these individual

[20] A more limited question is whether one can associate to each allocation an order on the set of agents reflecting how well they are treated. At an efficient allocation, the no-envy relation is acyclic but not transitive unless a strong assumption of similarity of preferences is made (Feldman and Weiman, 1979).

sums can be weighted. Varian (1976) suggests a particular way in which the weights can be made to depend on the allocation.

These measures all have the advantage of yielding complete rankings of the set of feasible allocations, although the ranking provided by the counting measure still has large indifference classes.[21] The measures of aggregate envy proposed by Feldman and Kirman (1974) do provide fine rankings of the set of allocations at which someone envies someone else, but all allocations at which no one envies anyone belong to the same indifference class.

The extent to which an agent envies another agent can be measured in other ways, and a fine ranking of all allocations derived from this measure, as follows: Given $z \in Z(e)$ for which there is a pair $\{i, j\} \subseteq N$ such that $z_j \; P_i \; z_i$, let $\lambda_{ij}(z) \in \mathbb{R}_+$ be the factor by which z_j should be reduced so as to obtain a bundle that agent i finds indifferent to z_i: $\lambda_{ij}(z)z_j \; I_i \; z_i$. Then we rank allocations at which there is envy as a function of the quantity $\sum_{\{i,j\} \subseteq N; z_j P_i z_i} \lambda_{ij}(z)$ (Chaudhuri, 1985, 1986).

Conversely, in order to measure the extent to which an allocation may exceed the no-envy requirement, let $z \in Z(e)$ be an allocation for which there is a pair $\{i, j\} \subseteq N$ such that $z_i \; P_i \; z_j$, but this time, let $\lambda_{ij}(z)$ be the factor by which z_j should be expanded so as to recover indifference, that is, so that once again $\lambda_{ij}(z)z_j \; I_i \; z_i$. This is a natural generalization of the previous idea. We then evaluate z by the sum $\sum \lambda_{ij}(z)$. Alternatively, instead of evaluating an allocation z by summing the terms $\lambda_{ij}(z)$, we could focus on the largest coordinate of the vector $(\lambda_{ij}(z))_{i,j \in N}$. Choosing the allocation at which this coordinate is as small as possible is more likely to produce a fair distribution of envy across agents when envy cannot be avoided, and to even out distance from envy, when envy-free allocations exist. The formal definition is as follows:

Definition (Diamantaras and Thomson, 1990) Given $e \equiv (R, \Omega) \in \mathcal{E}^N$, and $\lambda \in \mathbb{R}$, let $F^\lambda(e) \equiv \{z \in Z(e)\text{: for each } \{i, j\} \subseteq N, \; z_i \; R_i \; \lambda z_j\}$. Let $\lambda(e) \equiv \inf\{\lambda\colon F^\lambda P(R) \neq \emptyset\}$ and $H(e) \equiv F^{\lambda(e)}P(e)$.

The existence of $\lambda(e)$ is guaranteed if indifference curves are transversal to the axes, (and holds for significantly broader domains of economies than the

[21]Feldman and Kirman assess the extent to which these criteria distribute welfare "evenly" across agents. As they have the form of a summation, it is not surprising that the answer may be unsatisfactory, as frequently occurs from maximization of utilitarian objectives. Another relevant contribution is Allingham (1977).

domain considered in this section). This definition both maximally strengthens no-envy when the concept is too permissive without losing existence, and minimally weakens it so as to recover existence when it is too strong. Like the no-envy notion itself, the solution it generates violates *Pareto-indifference*. Although it is quite selective, it is not *essentially single-valued*. Finally, it is not the case, as one could have hoped, that for each allocation z it selects, and for each $i \in N$, there is $j \in N$ such that $z_i = \lambda(e)z_j$ (Kolpin, 1991a).

The contractions and expansions underlying these definitions can be adapted so as to extend, or select from, other equity notions (Thomson, 1995c), and from them rankings of the set of feasible allocations can be derived (for the special case of agents with identical and homothetic preferences, Chaudhuri, 1986, establishes a relation to rankings based on income inequality). An application to income taxation is developed by Nishimura (2002).[22]

Other geometric operations are conceivable. For instance, if an agent envies some other agent, one could use the distance between that second agent's bundle to the lower contour set of the first agent at his assigned bundle (Chaudhuri, 1986).

All of the definitions may have intuitive appeal, but none has axiomatic foundations. It is the chief merit of the Fleurbaey and Maniquet program cited earlier that it shows how rankings of allocations can be derived from axioms.

4.6 Economies with a large number of agents

Next, we turn to economies with a large number of agents modelled as an atomless measure space. An important result for that case concerns no-envy: informally, if preferences are smooth and "sufficiently dispersed", the set of envy-free and efficient allocations is approximately equal to the set of equal-division Walrasian allocations.

We have seen that in economies with finitely many agents, the equal-division Walrasian solution is a subsolution of the no-envy and Pareto so-

[22] Another idea is to formulate iterative procedures to improve the equity "content" of allocations at every step. For the case of two goods and two agents, an attempt at such a formulation is made by Baumol (1982): it consists in identifying the extreme points of the envy-free set and in successively selecting "a middle zone" (in a manner that we will not specify). Philpotts (1983) studies this procedure and shows that it is incompatible with efficiency.

lution independently of the number of agents, and of course this inclusion remains true for infinite economies. We will state a theorem on the approximate equivalence of the two solutions in large economies that is reminiscent of the asymptotic equivalence of the core and of the Shapley value (Shapley, 1953) with the Walrasian solution. However, it occurs here under less general conditions. In particular, one of the most convenient methods of modeling economies of increasing size, namely replication, does not guarantee convergence of the no-envy and Pareto solution to the equal-division Walrasian solution.

Although the study of variable-population requirements will mainly be discussed in later sections, it is convenient to introduce here a basic one. Given a profile R of preferences, and a natural number $k \in \mathbb{N}$, we denote by $k * R$ a profile obtained by introducing, for each $i \in N$, $k-1$ agents with the same preferences as his. We say that it is a "k-replica of R." The notation $k * z$ designates the corresponding k-replica of z. The requirement is that if an allocation is chosen for some economy, then for each $k \in \mathbb{N}$, and each k-replica economy, its k-replica should be chosen:

Replication-invariance: For each $N \in \mathcal{N}$, each $(R, \Omega) \in \mathcal{E}^N$, each $z \in \varphi(R, \Omega)$, each $N' \supset N$, each $k \in \mathbb{N}$, and each $(R', \Omega') \in \mathcal{E}^{N'}$, if (R', Ω') is a k-replica of (R, Ω), then $k * z \in \varphi(R', \Omega')$.

The no-envy and Pareto solution is obviously *replication-invariant*. Now, starting from a finite economy admitting an envy-free and efficient allocation that is not an equal-division Walrasian allocation, one can easily construct an atomless economy with the same feature. However, if preferences are smooth and "sufficiently dispersed", any envy-free and efficient allocation is an equal-division Walrasian allocation. The first formal result of this kind is due to Varian (1976). To formalize the idea that preferences are dispersed, we work with a continuum of agents indexed by a parameter $t \in T \equiv]0, 1[$, supposing that there is a function $u: T \times \mathbb{R}^\ell_+ \to \mathbb{R}$ representing their preferences such that agent t's welfare from consuming the bundle z is $u(t, z)$.
Assumption A: The function u is continuous in both arguments and for each $t \in T$, $u(t, \cdot)$ is strictly concave.

Under Assumption A, if z is an envy-free allocation and t, t' are close, we expect $z(t)$ and $z(t')$ to be close evaluated by either $u(t, \cdot)$ or $u(t', \cdot)$, that is, agents with similar preferences are treated similarly. If in addition, z is efficient, then the bundles $z(t)$ and $z(t')$ themselves are close: the function

$z: T \to \mathbb{R}^\ell_+$ is continuous. The next result is based on the stronger assumption that z is in fact differentiable.

Theorem 4.2 (Varian, 1976) Domain: private goods; continuum of agents whose preferences can be represented by a function satisfying Assumption A. *If an envy-free and efficient allocation is a differentiable function, it is an equal-division Walrasian allocation.*

The tightness of Varian's assumptions is examined by Kleinberg (1980) and McLennan (1980). Kleinberg (1980) and Champsaur and Laroque (1981) specify assumptions on the primitives of an economy with a continuum of agents guaranteeing the Walrasian conclusion. They require strict monotonicity, strict convexity, and smoothness of preferences, and model an economy as a mapping from some space of parameter values (an open subset of a finite-dimensional Euclidean space) onto a space of preference relations. If this mapping satisfies sufficient continuity properties, the only envy-free and efficient allocations are equal-division Walrasian allocations. Mas-Colell (1987) also identifies conditions on the primitives guaranteeing the implication.

Now, recall that at a strictly envy-free allocation, each agent finds his bundle at least as desirable as the average of the bundles received by any subgroup of the other agents (Subsection 4.4). An interesting equivalence holds:

Theorem 4.3 (Zhou, 1992) Domain: private goods; measure space of agents; preferences are strictly monotonic and differentiable in the interior of commodity space. *An allocation is efficient and the set of agents each of whom prefers the average bundle of some group to his own bundle has measure zero, if and only if it is an equal-division Walrasian allocation.*

We also note that under standard assumptions on preferences, the equal-division core and the set of equal-income Walrasian allocations coincide (Vind, 1971).

4.7 Equity criteria for groups

Next, we propose criteria designed to evaluate the relative treatment of groups. First are extensions of the no-domination requirement. **No-domination of, or by, equal division for groups** says that no group of agents should receive on average at least as much as equal division of each good and more than equal division of at least one good (this is equivalent to requiring that no group should receive on average at most equal division of at least one good and less than equal division of at least one good).

Next is a generalization of the equal-division lower bound: suppose that each agent is given access to an equal share of the social endowment, and require of an allocation that no group of agents should be able, by redistributing among themselves the resources they collectively control, to make each of its members at least as well off, and at least one of them better off:

Definition Given $e \equiv (R, \Omega) \in \mathcal{E}^N$, the allocation $\boldsymbol{z \in Z(e)}$ **belongs to the equal-division core of** $\boldsymbol{e}$, written as $\boldsymbol{z \in C_{ed}(e)}$, if for each $G \subseteq N$, there is no list $(z'_i)_{i \in G} \in \mathbb{R}^{\ell G}_+$ such that $\sum_G z'_i = \frac{|G|}{|N|}\Omega$; $z'_G \; R_G \; z_G$; and for at least one $i \in G$, $z'_i \; P_i \; z_i$.

We have seen that if $|N| = 2$, an allocation meeting the equal-division lower bound is envy-free. However, if $|N| > 2$, an allocation in the equal-division core may not be envy-free (Kolm, 1972; Feldman and Kirman, 1974). In fact, this situation is the rule. Indeed, absence of envy implies that two agents with the same preferences are assigned bundles that are indifferent according to these preferences, but this property is certainly not met at each equal-division core allocation. For economies parameterized by their endowment profiles, the violations are typical (Green, 1972; Khan and Polemarchakis, 1978). An interesting exception are replica economies. At an equal-division core allocation of such an economy, two agents with the same preferences receive bundles that are indifferent according to these preferences (Debreu and Scarf, 1963).

We continue with inter-group requirements. An allocation $z \in Z(e)$ satisfies **no-domination for groups** if there is no pair $\{G, G'\}$ of subsets of N such that $\frac{\sum_G z_i}{|G|} \geq \frac{\sum_{G'} z_i}{|G'|}$: on a *per capita* basis, G should not receive at least as much of each good as G', and more of at least one good. We obtain variants of the definition by requiring $|G'| = |G|$ or $G' = N \backslash G$. Also, in the first case, we can require $G' \subseteq N \setminus G$.

To adapt to groups the no-envy criterion, we have several choices. First, we can consider what a group G could achieve by redistributing among its members what has been assigned to any other group, these resource being adjusted to take account of their relative sizes, and require that no such redistribution should make each member of G at least as well off as he was initially, and at least one of them better off (Vind, 1971; Varian, 1974). We introduce the formal definition in two steps. Given $e \equiv (R, \Omega) \in \mathcal{E}^N$, $z \in Z(e)$, and $G \subseteq N$, **the list $(z'_i)_{i \in G}$ G-dominates z in e** if $z'_G \, R_G \, z_G$, and for at least one $i \in G$, $z'_i \, P_i \, z_i$.

Definition Given $e \equiv (R, \Omega) \in \mathcal{E}^N$, the allocation **$z \in Z(e)$ is group envy-free for e** if there is no $(z'_i)_{i \in G} \in B(z)$ that G-dominates z, where

$$B(z) \equiv \{(z'_i)_{i \in G}\text{: there is } G' \subseteq N \text{ with } \sum_G z'_i = \frac{|G|}{|G'|} \sum_{G'} z_i\}.$$

Alternatively, we can insist on $|G'| = |G|$ and set $B(z) \equiv \{(z'_i)_{i \in G}\text{: there is } G' \subseteq N \text{ with } |G'| = |G| \text{ and } \sum_G z'_i = \sum_{G'} z_i\}$ (Varian, 1974).[23] If we want $G' = N \setminus G$, we set $B(z) \equiv \{(z'_i)_{i \in G}\text{:} \sum_G z'_i = \frac{|G|}{|N|-|G|} \sum_{N \setminus G} z_i\}$. Finally, the first two definitions can be weakened by requiring $G' \subseteq N \setminus G$. (The concept of a strictly envy-free allocation of Subsection 4.4 can be seen as a step towards the definitions just given.)

Further generalizations, along the lines of Kolm's generalization of no-envy (Subsection 4.4) can also be formulated. Equal-division Walrasian allocations satisfy all of them. Moreover, under replication, there is a sense in which the set of efficient allocations that are group envy-free converges to the set of equal-division Walrasian allocations (Varian, 1974; Section 8). We also have the following equivalences:

Theorem 4.4 (Kolpin, 1991b) Domain: private goods; locally non-satiated and strictly convex preferences. *In an economy replicated at least twice, any equal-division core allocation is envy-free. If the economy is replicated at least* $3|N| - 1$*-times, any such allocation is group envy-free. (When, in the definition of group no-envy, comparisons are restricted to groups of the same size, it suffices to replicate three times for this conclusion to hold.)*

For economies with a large number of agents modelled as a continuum, direct equivalence results between several of these definitions hold (Varian, 1974). The set of group envy-free (according to our primary definition, but

[23] In the case $|G'| = |G|$, we could simply reallocate among the members of G the specific bundles attributed to the members of that group, but then we would simply recover the no-envy concept.

even if only groups of the same size are compared) and efficient allocations coincide with the set of equal-division Walrasian allocations (Varian, 1974).

We conclude this subsection with a short discussion of non-convex preferences. Envy-free and efficient allocations may not exist then, as we saw, and it is natural to ask whether weakening no-envy to no-domination would help. The answer is no:

Theorem 4.5 (Maniquet, 1999) Domain: private goods; strictly monotonic, smooth, but not necessarily convex preferences. *There are economies in which all efficient allocations violate no-domination.*

Theorem 4.5 can be proved by means of a two-good and three-agent example. If $|N| = 2$, the solution that selects the allocation most preferred by one of the two agents among all allocations that the other finds indifferent to equal division, does satisfy the two requirements of the theorem.

4.8 Egalitarian-equivalence

Our next criterion is at the center of a family of solutions that constitute the main alternative to the families based on the permutation idea and on the agent-by-agent lower or upper bounds. It involves comparisons to "reference" allocations whose fairness could not be disputed since they are composed of identical bundles:

Definition (Pazner and Schmeidler, 1978a) Given $e \equiv (R, \Omega) \in \mathcal{E}^N$, the allocation $\boldsymbol{z \in Z(e)}$ **is egalitarian-equivalent for** $\boldsymbol{e}$, written as $z \in E(e)$, if there is $z_0 \in \mathbb{R}^\ell_+$ such that $z \; I \; (z_0, \ldots, z_0)$.

The fact that $(z_0, \ldots, z_0)$ is not feasible has been at the origin of some opposition to the concept, but the reference to hypothetical situations in the evaluation of an actual alternative is not unreasonable. In abstract social choice theory, axiomatic bargaining theory, apportionment theory, to name just a few examples, the desirability of an outcome for some situation is often evaluated by comparing it to choices made in reference economies that have a larger or smaller feasible set, involve a greater or smaller number of agents, or exhibit certain symmetries not actually present.

Some critics have also argued that the criterion might lead us to declare desirable an allocation associated with a reference bundle that none of the agents cares for (Daniel, 1978). But this criticism is misdirected since such an allocation would in general not be efficient, and we presumably will want to complement the criterion with efficiency. It would be equally easy to identify economies admitting envy-free allocations at which each agent is indifferent between his assignment and the zero bundle. Should we object to the no-envy concept on these grounds?

Nevertheless, objections can indeed be raised against egalitarian-equivalence, and we will present them after a study of its basic features. On the positive side, we will see in later sections that it enjoys a variety of desirable properties not satisfied by the no-envy concept, justifying the important place it has in this exposition.

The existence of egalitarian-equivalent and efficient allocations is obtained under weak assumptions. Note in particular that in the following theorem, no convexity assumption is made:

Theorem 4.6 (Pazner and Schmeidler, 1978a) Domain: private goods; strictly monotonic preferences. *Egalitarian-equivalent and efficient allocations exist.*

The egalitarian-equivalence solution and its intersection with the Pareto solution both satisfy *Pareto-indifference* and *equal treatment of equals.* If $|N| = 2$, envy-free allocations are egalitarian-equivalent. This relation, however, fails for $|N| > 2$. In fact, an equal-division Walrasian allocation may then not be egalitarian-equivalent.

Conversely, an egalitarian-equivalent allocation may violate no-domination, and *a fortiori*, no-envy. In fact, extreme violations of no-domination may occur: at an egalitarian-equivalent and efficient allocation, a particular agent may receive the entire social endowment.[24] This suggests that the egalitarian-equivalence and Pareto solution is "too large". Are there selections from it that satisfy no-domination? The following paragraphs clarify the extent to which these various ideas can be reconciled.

Definition Let $r \in \mathbb{R}^{\ell}_{+}$, $r \neq 0$, be given. Given $e \equiv (R, \Omega) \in \mathcal{E}^N$, the allocation $\boldsymbol{z \in Z(e)}$ **is** $\boldsymbol{r}$**-egalitarian-equivalent for** $\boldsymbol{e}$, written as $\boldsymbol{z \in E_r(e)}$, if it is egalitarian-equivalent with a reference bundle proportional to r.

[24] Corchón and Iturbe-Ormaetxe (2001) propose a general definition of fairness based on expectations. It covers various notions discussed in the foregoing pages, such as no-envy and egalitarian-equivalence, as special cases.

The existence of r-egalitarian-equivalent and efficient allocations is guaranteed under the assumptions of Theorem 4.6.

For each $r \in \mathbb{R}^{\ell}_{+}\backslash\{0\}$, let $\bar{E}_r P$ be the selection from the egalitarian-equivalence and Pareto solution defined by requiring the reference bundle z_0 to be such that the vector $z_0 - \frac{\Omega}{|N|}$ is proportional to r, and let $\bar{E}P \equiv \bigcup_{r\in\mathbb{R}^{\ell}_{+}\backslash\{0\}} \bar{E}_r P$. The subcorrespondence of the egalitarian-equivalence and Pareto solution so obtained is a subsolution of the equal-division lower bound solution.

Obviously, if r and r' are proportional, $E_r = E_{r'}$. If $|N| = 2$ and r is proportional to Ω (but only then), the solution $E_r P$ satisfies no-domination. In fact, still if $|N| = 2$, if r is proportional to Ω and additionally preferences are convex, any r-egalitarian-equivalent and efficient allocation is envy-free. Without convexity, this inclusion may fail (Figure 3a).

If $|N| > 2$, requiring r to be proportional to Ω does not guarantee no-domination, although it obviously guarantees that the equal-division lower bound is met (Figure 3b).[25] More seriously, there are economies where all egalitarian-equivalent and efficient allocations violate no-domination.[26] Therefore, egalitarian-equivalence is fundamentally incompatible with no-domination (and therefore no-envy).

To compensate for this limitation, the egalitarian-equivalence correspondence enjoys a variety of appealing properties. In particular, it admits selections that are monotonic with respect to changes in resources, technologies, and other parameters, and we will come back to it on many occasions (for instance, Section 7). Moreover, it can be generalized so as to yield other concepts that also enjoy these properties, as we will discover at various points in this survey.

Conditions are known that guarantee the existence of Ω-egalitarian-equivalent and efficient allocations for economies with a large number of agents modelled either as an infinitely countable set or as a continuum (Sprumont and Zhou, 1999).

[25] Pazner and Schmeidler (1978a) provide other reasons for requiring the reference bundle to be proportional to the average bundle.

[26] This follows directly from an example constructed by Postlewaite and described by Daniel (1978) in order to establish the existence of economies where all egalitarian-equivalent and efficient allocations violate no-envy.

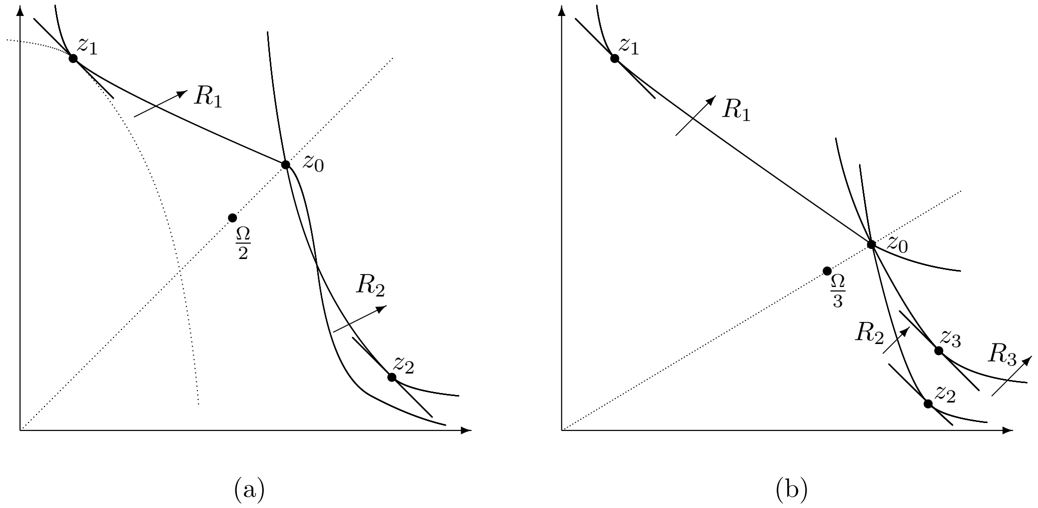


Figure 3: Clarifying the relation between no-domination, no-envy, and egalitarian-equivalence. (a) Here, $N = \{1, 2\}$. The dotted concave curve is the symmetric image of agent 2's indifference curve through z_2 with respect to equal division. The allocation z is r-egalitarian-equivalent and efficient for the two-agent economy depicted here, with $r \equiv \Omega$, but since agent 1 prefers z_2 to z_1, z is not-envy-free. (b) Here, $N = \{1, 2, 3\}$. The allocation z is r-egalitarian-equivalent and efficient for this three-agent economy, but since $z_3 > z_2$, z violates no-domination.

4.9 Equitable trades

Consider a society where each agent starts out with a bundle on which he has particular rights. The allocation defined by these individual endowments not being in general efficient, the problem arises as to how to distribute the gains made possible by exchanges among agents. In other words, how should we evaluate the process that takes an economy from its initial position to its final position when it is felt that the possibly uneven rights that agents have on resources should be taken into account?[27] Recall that an economy with individual endowments is a pair $e \equiv (R, \omega) \in \mathcal{R}^N \times \mathbb{R}_+^{\ell N}$. Let $\mathcal{E}_{end}^N$ be a domain of such economies. Given $e \in \mathcal{E}_{end}^N$, let $T(e) \subseteq \mathbb{R}^{\ell N}$ be its set of feasible trade profiles: $T(e) \equiv \{t \in \mathbb{R}^{\ell N} \colon \sum t_i = 0\}$.

We start with the evaluation of the equity of trades. The simplest idea is formulated in physical terms: the trade of no agent should dominate, good by good, that of any other agent:

Definition Given $e \equiv (R, \omega) \in \mathcal{E}_{end}^N$, the trade profile $\boldsymbol{t \in T(e)}$ **satisfies no-domination for** $\boldsymbol{e}$ if $(\omega + t) \in Z(e)$ and for no pair $\{i, j\} \subseteq N$, $t_j \geq t_i$.

Next is the counterpart of the equal-division lower bound. It is a standard requirement:

[27]Holcombe (1983) emphasizes this aspect of fairness.

Definition Given $e \equiv (R, \omega) \in \mathcal{E}^N_{end}$, the trade profile $\boldsymbol{t} \in \boldsymbol{T(e)}$ **meets the individual-endowments lower bound for** $\boldsymbol{e}$ if $(\omega + t) \in Z(e)$ and $(\omega + t)\ R\ \ \omega$.

The following concept is the counterpart for trades of the no-envy concept introduced earlier for allocations: if given access to any of the trade vectors $(t_j)_{j \in N}$, each agent $i \in N$ would choose the vector t_i intended for him:

Definition (Kolm, 1972; Schmeidler and Vind, 1972) Given $e \equiv (R, \omega) \in \mathcal{E}^N_{end}$, the trade profile $\boldsymbol{t} \in \boldsymbol{T(e)}$ **is envy-free for** $\boldsymbol{e}$ if $(\omega + t) \in Z(e)$ and for no pair $\{i, j\} \subseteq N$ such that $(\omega_i + t_j) \in \mathbb{R}^\ell_+$, we have $(\omega_i + t_j)\ P_i\ (\omega_i + t_i)$.

Obviously, no-envy implies that two agents with the same endowments and the same preferences receive bundles that are indifferent according to these preferences. The Walrasian solution satisfies this requirement, which generalizes *equal treatment of equals*. However, both the "cardinal value" and the "ordinal value", solutions induced on the economies under consideration here by the Shapley value (Shapley, 1953) violate it (Yannelis, 1985).

Imagine now that agents have repeated access to any of the trades in the list $(t_i)_{i \in N}$. We obtain the following stronger definition, which embodies a requirement of anonymity:

Definition (Schmeidler and Vind, 1972) Given $e \equiv (R, \omega) \in \mathcal{E}^N_{end}$, the trade profile $\boldsymbol{t} \in \boldsymbol{T(e)}$ **is strongly envy-free for** $\boldsymbol{e}$ if $(\omega + t) \in Z(e)$, and for no $i \in N$ and no $\alpha \in \mathbb{N}^N$, $(\omega_i + \sum \alpha_j t_j)\ P_i\ (\omega_i + t_i)$.

Walrasian trades are easily seen to be strongly envy-free. More interesting is that under weak conditions on preferences, a converse holds (Schmeidler and Vind, 1972).

The no-envy concept can also be adapted to evaluate the trades of groups. Given $z \in Z(e)$, we could require that there should be no pair of groups $\{G, G'\} \subseteq N$ with $G \cap G' = \emptyset$ and a list $t \equiv (t_i)_{i \in G} \in \mathbb{R}^{\ell|G|}$ of trades for the members of G such that $\sum_G t_i = \sum_{G'} (z_j - \omega_j)$; $(\omega + t)\ R_G\ z$; and for at least one $i \in G$, $(\omega_i + t_i)\ P_i\ z_i$. Note that in this definition, the relative sizes of the two groups are not restricted. The set of allocations passing this test is contained in the core and it contains the set of Walrasian allocations (Jaskold-Gabszewicz, 1975; Yannelis, 1985). The extent to which an allocation obtained through a group envy-free trade differs from a Walrasian allocation can be quantified. Yannelis (1985) proves the existence of allocations meeting an approximate version of group no-envy in economies with small and large traders in which small traders may not have convex preferences.

Jaskold-Gabszewicz (1975) also considers economies with both atoms and an atomless sector and studies this notion of envy-free trades for groups, when the admissible groups are arbitrary, and alternatively, when they are restricted to be subsets of the two sectors. He relates the sets of allocations satisfying these definitions to the core and to the set of Walrasian allocations. If there are no atoms, the core coincides with the set of group envy-free allocations. Also, at a group envy-free allocation of such a mixed economy, and even if only groups of equal measures are considered in the definition, the value of the trades of all small agents are equal (Shitovitz, 1992).

4.10 Towards a complete theory of equity

Suppose that it has been decided that gains from trade should be distributed according to a particular method. One could hope that the equity of the final allocation would be guaranteed if the initial allocation is itself equitable. But how is one to judge the equity of an initial allocation? A complete theory of equity would involve choosing criteria for initial positions, final positions, and trades. Now, the question is whether the criteria chosen for trades and initial allocations should bear any relation to the criterion chosen for allocations, or whether these choices can be made independently. In this subsection, we attempt to lay the foundation for an integrated theory of fairness.[28]

We start with the evaluation of initial allocations. Once again, equal division is an appealing choice. In fact, equal division seems to be a more legitimate initial-state than end-state principle since its (usual) inefficiency will hopefully be removed by any reasonable transition principle. However, the end-state principles discussed in Section 4 can be used as initial-state principles too. For instance, we can certainly define an envy-free *initial* allocation as we defined an envy-free *final* allocation.

[28]For a discussion of related issues, see Varian (1975). Yamashige (1997) can be seen as a contribution to this program too, as he proposes the following test on a transition principle φ: if agent j's endowment dominates agent i's endowment, commodity by commodity, then at the allocation chosen by φ, agent j should not envy agent i. For consistency, one could argue that the same test should be performed on initial and final allocations, and for greater generality, base the comparisons on other fairness notions.

Next, we submit that there are good reasons to preserve the conceptual distinction between initial-state and end-state principles. Elaborating on the previous example, the fact that an agent would not want to *consume* anyone else's initial bundle may be thought irrelevant; the initial position matters only to the extent that it affects the final position. In more general situations, preferences may not even be defined on initial positions: for instance, in a production economy, the initial position would commonly be given by the specification of how much of each of the factors of production (machinery, land, ...) each agent controls. These resources cannot be directly consumed but their initial distribution will in general affect the final allocation.

The following facts, which have been described in the literature as "paradoxes", clearly indicate that certain consistency conditions should indeed be respected.

We have already seen that (i) if $|N| \geq 3$, starting from equal division (an envy-free but typically not efficient allocation), a trade to the core may lead to an allocation that is not envy-free (Kolm, 1972; Feldman and Kirman, 1974). In fact, starting from equal division, a sequence of envy-free trades may lead to a core allocation that is not envy-free (Golman and Sussangkarn, 1980). (ii) Also, and even if $|N| = 2$, starting from an envy-free (and not efficient) allocation, an envy-free trade profile (even a Walrasian trade profile) may lead to an allocation that is not envy-free (Kolm, 1972; Feldman and Kirman, 1974). (iii) Moreover, and again even if $|N| = 2$, starting from an envy-free (and non-efficient) allocation, there may be no Pareto-improving trade leading to an envy-free and efficient allocation (Goldman and Sussangkarn, 1978). Since, in general, there are envy-free trades that are not Pareto improving, the following fact is even more serious. (iv) Even if $|N| = 2$, starting from an envy-free (and not efficient) allocation, there may be no envy-free trade leading to an envy-free and efficient allocation (Thomson, 1982).

The complex structure of the set of envy-free allocations is reflected in its image in utility space, but at least this image is connected, which is not the case for the former set.

The main lesson to be drawn from (i)-(iv) is that it is not legitimate to arbitrarily and independently select initial-state, transition, and end-state principles. These choices should be linked in some way. We will decompose the search for links by first assuming that a transition principle has been selected, and in showing that on the basis of this choice, natural restrictions on both the initial-state and the end-state principles can be formulated. Then, we will suggest how transition principles can in turn be derived from end-state principles. We will conclude by requiring the "consistency" of the two operations. (These developments follow Thomson, 1983a, and Thomson and Varian, 1985.)

4.10.1 Deriving initial state and end-state principles from transition principles

The derivation of initial-state and end-state principles from transition principles is achieved by a simple extension of the permutation idea. Let φ be a transition principle:

Definition Given $e \equiv (R, \omega) \in \mathcal{E}^N_{end}$, the pair $(\omega, z) \in Z(e) \times Z(e)$ of an initial allocation and a final allocation is a **φ-acceptable pair for e** if for each $\pi \in \Pi^N$, $z \in \varphi(R, \pi(\omega))$.

Here, the initial position is evaluated indirectly, via its effect on the final allocation. Similarly, the final allocation is evaluated indirectly, by checking its independence from permutations of the components of the initial position.

Since equal division is not affected by permutations, it is clear that, as soon as φ is well-defined, φ-acceptable pairs exist. Indeed, any pair of the form $(\bar{\omega}, z)$, where $\bar{\omega} \equiv (\frac{\sum \omega_i}{|N|}, \ldots, \frac{\sum \omega_i}{|N|})$ and $z \in \varphi(R, \bar{\omega})$, satisfies the definition.

However, the free choice of ω may lead to a violation of no-domination, and if we insist on this important requirement, the consistency requirement suggested above should be strengthened. We propose to strengthen it by placing a natural restriction on the class of comparisons in which society engages in order to decide whether an allocation is equitable, (and limiting our attention to subsolutions of the Pareto solution). Assuming a commitment to efficiency, and given an efficient allocation whose equity is to be evaluated, we declare its comparison to non-efficient allocations irrelevant. Since the allocation $\pi(z)$ obtained from some z through an arbitrary permutation π is in general not efficient, we ignore it as a possible candidate.

It remains that permuting the components of z is appealing. The question then is whether one could associate with $\pi(z)$ some efficient allocation that is equivalent to it from the viewpoint of equity and to which z could be compared instead. We suggest that if an equitable transition principle has been adopted, this can be done since whatever inequities exist in $\pi(z)$ will presumably be preserved by its operation. This line of thought leads us to the following test: starting from some efficient allocation z, we permute its components to obtain $\pi(z)$, and we reestablish efficiency by operating φ. It is to the resulting allocation(s) that we compare z. We require that this process returns to z, or rather *can* return to z.

Definition Given $e \equiv (R, \omega) \in \mathcal{E}^N_{end}$, the allocation $\boldsymbol{z \in Z(e)}$ **is** $\boldsymbol{\varphi}$**-acceptable for** $\boldsymbol{e}$ if for each $\pi \in \Pi^N$, $z \in \varphi(R, \pi(z))$.

Let us apply the definitions to some examples:

Theorem 4.7 (Thomson, 1983a) Domain: private goods; monotonic and convex preferences. *The acceptable allocations relative to* (a) *the individual-endowments lower bound and Pareto transition correspondence are the envy-free and efficient allocations;* (b) *the Walrasian transition correspondence are the equal-division Walrasian allocations.*

4.10.2 Deriving transition principles from end-state principles.

Summarizing the progress we have made, our problem of selecting initial-state, transition, and end-state principles has been reduced to the selection of a transition principle. In Subsection 4.9 we already suggested several criteria to evaluate trades, but we would like to be systematic and coherent in our choice.

Could *internal* considerations such as the ones that helped us derive end-state principles from transition principles permit us to achieve this objective? We now argue that indeed such considerations can be brought in to *derive transition principles from end-state principles*. First, recall the definition of an envy-free trade (Subsection 4.9): this is a feasible trade $t \in \mathbb{R}^{\ell N}$ such that for each $i \in N$ and each $\pi \in \Pi^N$, if $(\omega_i + \pi_i(t)) \in \mathbb{R}^{\ell}_{+}$, then $z_i \equiv (\omega_i + t_i)\ R_i\ (\omega_i + \pi_i(t))$. This transition principle is obtained from the end-state principle of an envy-free allocation by substituting trades for final consumption bundles, and using the preference relations on trades induced in the natural way from the preference relations on final consumption bundles. We propose to apply this transformation generally, thereby defining a mapping from end-state principles to transition principles that mirrors the mapping from transition principles to end-state principles defined in the previous section. Examples of transition principles that can be so obtained are the notion of an **average-envy-free trade for $(\boldsymbol{R}, \boldsymbol{\omega})$**, that is, a trade $t \in \mathbb{R}^{\ell N}$ such that for no $i \in N$, $(\omega_i + \frac{\sum_{N\setminus\{i\}} z_j}{|N|-1})\ P_i\ (\omega_i + t_i)$, and the notion of an **egalitarian-equivalent trade for $(\boldsymbol{R}, \boldsymbol{\omega})$**, that is, a trade $t \in \mathbb{R}^{\ell N}$ such that for some $t_0 \in \mathbb{R}^{\ell}$, and for each $i \in N$, $(\omega_i + t_i)\ I_i\ (\omega_i + t_0)$.

4.10.3 Consistency between transition principles and end-state principles

We are now ready to close the loop. Starting from some end-state principle, we derive from it a transition principle as just indicated. From this transition principle, we derive the end-state principle as explained earlier; **the consistency test that we suggest is that we get back to where we started.** Are there end-state principles such that the loop closes back on itself?

The answer is yes, and we possess all the elements to provide it. If the end-state principle is the equal-division Walrasian solution, then the transition principle is the (standard) Walrasian solution, which takes us back to the equal-division Walrasian solution as end-state principle.

To give an example of a sequence that does not fold back on itself, suppose that the end-state principle is the equal-division lower bound. Then, the derived transition principle is Pareto-domination, which in turn leads us to the end-state principle of no envy for allocations.

Usually, the loop will indeed not close back on itself. The "fixed point" property satisfied by the pair {Walrasian solution, equal-division Walrasian solution} does not seem to be shared by many other examples.

Another way of relating initial allocations, trades, and final allocations is proposed by Maniquet (2001): the transition principle should be **robust under partial delivery** of the trades it specifies. If a trade $t \in \mathbb{R}^{\ell N}$ is chosen for some economy (R, ω), and an arbitrary proportion α of this trade is carried out, then the remainder of it, $(1 - \alpha)t$, should be chosen for the revised economy $(R, \omega + \alpha t)$.[29] On the domain of economies with individual endowments in which preferences are smooth, if a subsolution of the no-envy in trades and Pareto solution is *robust under partial delivery*, then in fact it is a subsolution of the Walrasian solution.

[29] Maniquet calls this invariance property "decomposability".

5 Economies with production

Although we have been concerned up to now only with the fair allocation of a *fixed* bundle of goods among agents with equal rights on them, another fundamental issue is that of fair allocation when agents have contributed differently to the production of these goods, because they have supplied unequal amounts of their time, or because they have unequal productivities, or both.

A number of results described next pertain to this more general situation, in which some inputs are "agent-specific". The consumption bundle of each agent $i \in N$ has a coordinate representing his consumption of leisure, ℓ_i. Through an appropriate normalization, we assume that each agent i's maximal consumption of leisure is 1, so that his labor input is $1-\ell_i$. If skills differ across agents, labor inputs have to be distinguished according to who supplies them. Then, production possibilities are given as a subset Y of $\mathbb{R}^{n+m}$, where n is the number of labor inputs (equal to the number of agents) and m is the number of produced goods.

Let $\mathcal{Y}$ be a class of admissible production sets. An economy is now a pair $(R, \Omega, Y) \in \mathcal{R}^N \times \mathbb{R}^\ell_+ \times \mathcal{Y}$. Let $\mathcal{E}^N_{pro}$ be our generic notation for a class of such economies.

5.1 Adapting the basic concepts

A first way to extend the notion of an envy-free allocation to such economies is by having each agent $i \in N$ compare his *complete* consumption bundle (ℓ_i, x_i) to those of the other agents. Unfortunately, we run into the fundamental incompatibility with efficiency:

Theorem 5.1 (Pazner and Schmeidler, 1974) Domain: private goods; strictly monotonic and linear preferences; linear production function with agent-specific inputs. *Envy-free and efficient allocations may not exist.*

Interestingly, at an efficient allocation of an economy with production, two agents may mutually envy each other, a situation that, as the reader may recall, cannot occur in an exchange economy (Proposition 4.1).

We emphasize that in Theorem 5.1, preferences are quite well-behaved. It is true that if either all abilities or all preferences are the same, envy-free and efficient allocations do exist (Varian, 1974), but these assumptions are of course overly restrictive. Now, consider a two-good economy, the two goods being time and another good produced from labor. Suppose that agents can be ordered by their productivities and that their preferences satisfy the following Spence-Mirrlees "single-crossing" condition: given any two agents, at each bundle in their common consumption space, the increase in the consumption good required to keep a less productive agent on the same indifference curve when he gives up an arbitrary amount of leisure is greater than the corresponding increase for the more productive agent. Then, if the production technology is linear, envy-free and efficient allocations exist (Piketty, 1994).

Finally, in a production economy in which agents differ in their productivities, there may be no efficient allocation at which each agent finds his bundle at least as desirable as the average of the bundles received by all agents (a criterion proposed by Pazner, 1977). This can be seen by means of the example Pazner and Schmeidler (1978a) use to prove Theorem 5.1, because in that example, preferences are convex.[30]

Faced with the fundamental negative result stated as Theorem 5.1, a number of authors have proposed alternative definitions of equity. We have already encountered egalitarian-equivalence and balanced envy (Subsection 4.4). Egalitarian-equivalent and efficient allocations exist quite generally (Pazner and Schmeidler, 1978a). The main assumption is "welfare-connectedness": if an agent consumes a non-zero bundle, resources can be redistributed away from him so that all other agents are made better off. Under sufficiently strong convexity assumptions on preferences and technologies, the reference bundle in the definition of egalitarian-equivalence can be required to be proportional to the average consumption bundle and existence preserved. The assumption under which balanced-envy and efficient allocations are known to exist are more restrictive (mainly, that the production set is a convex cone, and that no two Pareto-efficient allocations be Pareto-indifferent; Daniel, 1975).

The following proposal is another generalization of no-envy. It recognizes the envy of agent $j \in N$ by agent $i \in N$ only after agent i's consumption of leisure is adjusted for him to produce the output produced by agent j:

Definition (Varian, 1974) Given $e \equiv (R, \Omega, Y) \in \mathcal{E}^N_{pro}$, the allocation

[30]Pazner refers to it as "per-capita fairness". Similarly, since average no-envy and the various criteria of group equity coincide with one of these criteria for $|N| = 2$, we conclude that existence also fail for them in production economies with differentially productive agents.

$\boldsymbol{z} \in \boldsymbol{Z}(\boldsymbol{e})$ **is productivity-adjusted envy-free**[31] **for** $\boldsymbol{e}$ if for each $i \in N$, $(\ell_i, x_i)\ R_i\ (\ell_i(\bar{x}_j), x_j)$, where $\bar{x}_j$ is the bundle produced by agent $j \in N$, and $1 - \ell_i(\bar{x}_j)$ is the amount of labor that agent $i \in N$ needs to produce $\bar{x}_j$.

The concept has the technical disadvantage of being well-defined only if the production set is additive, since only then is it possible to identify the share of the total output produced by each agent. Another, more fundamental problem with it is that, in a sense, it lets agents with high productivity appropriate the benefits of their greater skills. To the extent that higher productivity is "earned", through a lengthier education or greater exertion on the job, this may be legitimate (but would be recognized in a model in which all inputs are properly included in the description of an allocation). However, if it is the result of greater innate ability, one may well object to it (Pazner, 1977; see below).

A proof of the existence of productivity-adjusted envy-free and efficient allocations can be given along the lines of the "Walrasian" proof of existence of envy-free and efficient allocations in exchange economies and under the same assumptions. It suffices to operate the Walrasian solution from equal division of the produced goods but leaving to each agent the ownership of his time endowment. At any allocation reached in this manner, each agent produces a bundle whose value at the equilibrium prices is equal to the value of the bundle produced by each other agent (Varian, 1974).

The intent of the next concept is to distribute across agents the benefits derived from the greater productivity of the more productive among them (Pazner and Schmeidler, 1978b; Varian, 1974; Pazner, 1977). The solutions using either equal division as a lower bound on welfares or no-envy are not easily adaptable because the preferences of an agent are not defined on a space that includes other agents' leisure, but one can at least take advantage of the instrumental value of the Walrasian solution in delivering envy-free allocations when there is no production, and in providing equal opportunities. Let us then operate the Walrasian solution from equal division of all goods, including time endowments.[32] Svensson (1994b) states an existence result for allocations at which implicit incomes are equal, and neither preferences nor technology are necessarily convex. As in the extensions of Theorem 4.1, where

[31] Varian (1974) refers to these allocations as "wealth-fair".

[32] These authors speak of the equalization of "implicit incomes", or "potential incomes", or "full incomes", and use the term "income-fairness" for the resulting fairness notion.

the main assumption pertains to the topological structure of the Pareto set, his critical assumption is that the Pareto set be invariant under replication. (Section 8 is devoted to an analysis of properties of this type.)[33]

A concept in the same spirit as that of a productivity-adjusted envy-free allocation, but subject to the same limitations, is formulated by Otsuki (1980). Biswas (1987) also defines an equity notion in production economies with differently productive agents.

Non-convexities in technologies present another difficulty for the existence of envy-free and efficient allocations:

Proposition 5.1 (Vohra, 1992) Domain: private goods; strictly monotonic and convex preferences; no agent-specific inputs; not necessarily convex technologies. *Envy-free and efficient allocations may not exist.*

In the face of this negative result, which in fact can be proved by means of an example in which in fact the only source of non-convexity in the technology is the presence of a fixed cost, Vohra proposes to weaken no-envy by imposing a certain symmetry among all agents with respect to possible occurrences of envy (a notion related to one suggested by Varian, 1974):

Definition (Vohra, 1992) Given $e \equiv (R, \Omega, Y) \in \mathcal{E}^N_{pro}$, the allocation $\boldsymbol{z} \in$ $\boldsymbol{Z(e)}$ **is essentially envy-free for** $\boldsymbol{e}$ if for each $i \in N$, there is $z^i \in Z(e)$ that is Pareto-indifferent to z and at which agent i envies no one.

If preferences are strictly convex, this definition reduces to no-envy. This weakening of the standard definition suffices to recover existence. In fact, existence holds without any convexity assumption on either preferences or technologies. A critical one, however, remains that there be no agent-specific input:

Theorem 5.2 (Vohra, 1992) Domain: private goods; strictly monotonic preferences; no agent-specific input; non-empty and compact feasible set. *Essentially envy-free and efficient allocations exist.*

[33] As a compromise between the two definitions just discussed, Archibald and Donaldson, 1979, propose that each agent be given ownership of a certain fraction of his own time and an equal share of the remaining time of everyone. At equilibrium, budget sets are not related by inclusion, which they suggest is a minimal requirement of “fairness of implicit opportunities” (Section 6).

In economies in which no two distinct Pareto-efficient allocations are Pareto-indifferent, an allocation is essentially envy-free as well as efficient if and only if it is envy-free and efficient. In general, if $|N| > 2$, the set of allocations that are Pareto-indifferent to an envy-free and efficient allocation (these allocations may violate no-domination) is a strict subset of the set of essentially envy-free allocations.

5.2 Agent-by-agent lower and upper bounds

No-envy and egalitarian-equivalence notions are based on interpersonal comparisons of bundles. We now turn to criteria that, by contrast, can be evaluated agent by agent, just like the equal-division lower bound.

First, for each agent, we imagine an economy composed of agents having preferences identical to his, and we identify his welfare under efficiency and *equal treatment of equals*. We take this welfare as a bound. For nowhere-increasing returns-to-scale, the profile of these welfares is feasible, and it can be used to define a lower bound requirement on welfares.

Definition (Gevers, 1986; Moulin, 1990d) Given $e \equiv (R, \Omega, Y) \in \mathcal{E}^N_{pro}$, the allocation $z \in Z(e)$ meets the **identical-preferences lower bound for e** if for each $i \in N$, $z_i \; R_i \; z^*_i$, where z^*_i is a bundle that agent i would be assigned by any efficient solution satisfying *equal treatment of equals* in the hypothetical economy in which each other agent had preferences identical to his.[34]

Alternatively, we could imagine each agent in turn to have control of an equal share of the social endowment and unhampered access to the technology:[35]

Definition (Moulin, 1990d) Given $e \equiv (R, \Omega, Y) \in \mathcal{E}^N_{pro}$, the allocation $z \in Z(e)$ meets the **equal-division free-access upper bound for e** if for each $i \in N$, $z^*_i \; R_i \; z_i$, where z^*_i is a bundle that maximizes agent i's preferences if given access to $(\frac{\Omega}{|N|}, Y)$.

[34]Gevers uses the phrase "egalitarian lower bound" and Moulin the phrase "unanimity lower bound". The bound proposed by Steinhaus (1948) can also be understood in this way (Section 12).

[35]Yoshihara (1998) proposes an extension of this notion to the case of economies with arbitrarily many goods.

This definition can be generalized by imagining each group of agents in turn to have control over a proportion of the social endowment equal to its relative size in the economy and unhampered access to the technology:

Definition (Foley, 1967) Given $e \equiv (R, \Omega, Y) \in \mathcal{E}^N_{pro}$, the allocation $z \in Z(e)$ is in the **equal-division free-access core of e** if there is no $G \subseteq N$ and no list of bundles $(z_i^*)_{i \in G} \in \mathbb{R}^{\ell G}_+$ such that for each $i \in G$, $z_i^*\ R_i\ z_i$, and for at least one $i \in G$, $z_i^*\ P_i\ z_i$, this list being attainable by the group G if given access to $(\frac{|G|}{|N|}\Omega, Y)$.

What of the compatibility of these bounds with no-envy? First, we state an incompatibility:

Proposition 5.2 (Moulin, 1990c) Domain: one-input one-output production economies; monotonic and convex preferences; concave production function. *There are economies in which no allocation that is envy-free and efficient meets the equal-division free-access upper bound.*

The equal-division free-access upper bound itself is met on the domain of the theorem by selections from the Pareto solution, in particular by a solution we find more convenient to define later, the constant-returns-to-scale–equivalent solution (Mas-Colell, 1980a; Moulin, 1987b) (See the discussion following Theorem 7.3).

In the case of nowhere-decreasing returns-to-scale, the equal-division free-access bound becomes a lower bound, and an impossibility parallel to that stated in Proposition 5.2 obtains: no subsolution of the Pareto solution satisfies no-envy for trades and meets the equal-division free-access lower bound (Moulin, 1987b).

When preferences are quasi-linear, the welfare of a group at an allocation can be defined as the aggregate utility of its members evaluated by means of the quasi-linear representations. Then, the core constraints form a system of inequalities that is familiar in the theory of transferable utility coalitional games.

Next is another bound for one-input one-output production economies. Say that an allocation z is a **proportional allocation for e** if there are prices such that each agent i maximizes his preferences given these prices at z_i. Let $Pro(e)$ be the set of these allocations. An allocation z satisfies the **constant-returns-to-scale lower bound** if for each $\bar{z} \in Pro(e)$, $z \; R \; \bar{z}$ (Maniquet, 1996).[36] Such allocations exist very generally, but on the domain of economies with concave production functions, no solution jointly satisfies the constant-returns-to-scale lower bound and the identical-preferences lower bound.

A "proportional equilibrium" is a configuration of input contributions that constitute a Nash equilibrium of the game that results when agents are paid for their input contributions according to average cost. The proportional equilibria can be Pareto ranked. We require next that **each agent should find his bundle at least as desirable as what he would receive at the Pareto superior proportional equilibrium**. The existence of allocations satisfying this bound can be established by standard arguments. Unfortunately, there may be no such allocation satisfying no-envy (Maniquet, 1996b).

Fleurbaey and Maniquet (1996a, 1999a) formulate yet other bounds and study their compatibility with other criteria (Chapter 21). They consider a model in which each agent is described in term of his preferences over a two-dimensional commodity space and a productivity parameter. The **constant-returns-to-scale lower bound** is defined for each agent by reference to the best bundle he could achieve if he had access to a constant-returns-to-scale technology, the same for all agents; the **work-alone lower bound** is defined for each agent by reference to the best bundle he could achieve if given unhampered access to the technology but under the obligation to provide bundles to the other agents to which he would not prefer his own.

For another study of the logical relations between bounds in a class of two-good economies with convex production sets, the identical-preferences lower bound and the free-access upper bound, see Watts (1999). She shows that, except in trivial cases, the latter does not imply the former.

A general theory of "aspirations" that encompasses several of the notions discussed above is developed by Corchón and Iturbe-Ormaetxe (2001).

[36]Maniquet uses the phrase "average-cost lower bound".

6 Equal opportunities as equal, or equivalent, choice sets

In this section, we switch our focus from allocations to the opportunities that lead to allocations. The notion of equal opportunities is of course central in the theory of economic justice, and many references could be given (Arneson, 1989; Cohen, 1989; Fleurbaey, 1995a, are representative pieces). We are interested here in implementing it in concrete economic environments, and exploring various notions of "equal opportunities as equal, or equivalent, choice sets." These notions formalize and generalize ideas informally discussed by a number of authors.

The phrase "equal opportunities" has been given a variety of meanings. When used in economies affected by uncertainty, it may mean "equal treatment ex-ante"; after the choice of nature is known, agents may end up with bundles that are not necessarily equitable, but redistributions may not be possible. Uncertainty may be endogenously generated by an allocation rule. Anticipating briefly on Section 10, consider the problem of allocating an indivisible good. A lottery giving all agents equal chances might be deemed equitable ex-ante although the final allocation may well appear unequitable.

Alternatively, in a context where agents' opportunities today are determined by decisions they made yesterday, equal opportunities may mean having access to the same set of decisions at that early stage. An example here is education. Giving two children with equal talents access to the same educational opportunities ensures that whatever differences exist between them later in life are due to their own decisions, for instance how hard they studied in school. It is often argued that, because of incentive considerations, we should not attempt to equalize end-results but should limit ourselves to giving people equal chances to develop their potential. Then, equal opportunities are provided by the transition mechanism (Subsection 4.9). Here, we ignore questions of uncertainty and incentives, and focus instead on comparing availability of concrete choices given in commodity space. This section is mainly based on Thomson (1994a).

Other approaches have been taken for the comparative evaluation of opportunity sets. One line of investigation was opened by Barberà, Barrett, and Pattanaik (1984) and pursued by Pattanaik and Xu (1992), Klemisch-Ahlert (1993), Bossert, Pattanaik, and Xu (1994), Kranich (1996), and Bossert (1997), to name a few representative contributions. These authors derive axiomatic characterizations of various rules to rank opportunity sets. (For a survey of this literature, see Peragine, 1999).

An initial limitation of this literature is that most of it was written for abstract domains, no account being taken of the natural restrictions on domains and alternatives that characterize concretely specified economic domains. Also, and for the same reason, they ignore restrictions on preferences that are natural on such domains. Recent contributions have addressed these issues however (Kranich, 1997; Xu, 2004).

6.1 Equal opportunities

Another way to give substance to the idea of equal opportunities is to let each agent choose his consumption bundle from a common choice set, as suggested in particular by Kolm (1973).

Although the straight-line choice sets of Walrasian analysis first come to mind, for a number of practical and theoretical reasons, we may want to consider other possibilities. First, even in economies where resources are supposedly allocated by operating the Walrasian mechanism, in practice, agents rarely face linear prices. Quantity discounts, quantity constraints, non-linear tax rates, welfare payments, all contribute to generating choice sets that depart considerably from standard Walrasian budget sets. Neither convexity of choice sets nor smoothness of their boundaries should be expected. In the theory of revealed preference, generalized notions of choice sets have been discussed and a complete treatment elaborated (Richter, 1979). Here too, we would like to start from abstract choice sets with no *a priori* restrictions. How generally can this be done?

The difficulty with giving all agents the same choice set is of course that the list of choices they will make from it will in general not constitute a feasible allocation: aggregate feasibility of the profile of choices precludes that the set be specified once and for all, before preferences are determined. Instead, one should have access to a "rich enough" family of choice sets, that is, a family such that, no matter what preferences are, it is guaranteed that for at least one member of the family, the list of chosen bundles constitutes a feasible allocation. In addition, one would prefer efficiency to hold whenever feasibility does. Although experience with the modified Walrasian budget sets agents face in the real world should make us doubt that this will be the case very generally, the search for families for which it does is certainly worthwhile. Let $\mathcal{B}$ be a family of choice sets (subsets of $\mathbb{R}^{\ell}_{+}$). We state the definitions for economies with production, $\mathcal{E}^N_{pro}$ being a generic domain of such economies with agent set N.

Definition Given $e \equiv (R, \Omega, Y) \in \mathcal{E}^N_{pro}$, the allocation $\boldsymbol{z} \in \boldsymbol{Z(e)}$ **is an equal-opportunity allocation relative to $\boldsymbol{\mathcal{B}}$ for $\boldsymbol{e}$**, written as $\boldsymbol{z} \in \boldsymbol{O_{\mathcal{B}}(e)}$, if there is $B \in \mathcal{B}$ such that for each $i \in N$, z_i maximizes R_i on B.

Definition The family $\boldsymbol{\mathcal{B}}$ is satisfactory on the domain $\mathcal{E}^N_{pro}$ if for each $e \in \mathcal{E}^N_{pro}$, $\emptyset \neq O_{\mathcal{B}}(e) \subseteq P(e)$.

A satisfactory family is easy to find: the **equal-income Walrasian family**, $\boldsymbol{\mathcal{W}_{ei}}$, is satisfactory on the classical domain.

It turns out that under natural assumptions on preferences and on the family $\mathcal{B}$, if $\mathcal{B}$ is satisfactory, then $O_{\mathcal{B}} \supseteq W_{ei}$. Families $\mathcal{B}$ exist for which $O_{\mathcal{B}} \supset W_{ei}$ however. For instance, if $|N| = 2$, families $\mathcal{B}$ can be defined such that $O_{\mathcal{B}} = \underline{B}_{ed}P$. Of course, for each family $\mathcal{B}$, we have $O_{\mathcal{B}} \subseteq F$. This inclusion is in fact a common justification for no-envy, namely an implication of equal opportunities, some say "equal liberty".

6.2 Equal-opportunity–equivalence

Another concept is obtained by generalizing the reasoning underlying the notion of egalitarian-equivalence. It involves checking whether an allocation under consideration is such that, for some member of the family of choice sets, each agent is indifferent between what he receives and the best bundle he could attain. Again, let $\mathcal{B}$ be a family of choice sets:

Definition (Thomson, 1994a) Given $e \equiv (R, \Omega, Y) \in \mathcal{E}^N_{pro}$, the allocation $\boldsymbol{z} \in \boldsymbol{Z(e)}$ **is equal-opportunity–equivalent relative to $\boldsymbol{\mathcal{B}}$ for $\boldsymbol{e}$**, written as $\boldsymbol{z} \in \boldsymbol{O^{\approx}_{\mathcal{B}}(e)}$, if there is $B \in \mathcal{B}$ such that for each $i \in N$, $z_i \; I_i \; z^*_i$, where z^*_i maximizes R_i on B.

If $\mathcal{B} \equiv \{\{z_0\}: z_0 \in \mathbb{R}^\ell_+\}$, then $O^{\approx}_{\mathcal{B}} = E$ (egalitarian-equivalence). If $|N| = 2$, and $\mathcal{B}$ is the family of all linear choice sets, then $O^{\approx}_{\mathcal{B}} = P$. For $|N| > 2$, there is no containment between $O^{\approx}_{\mathcal{B}}P$ and EP. If $\mathcal{B} \equiv \mathcal{W}_{ei}$, then $O^{\approx}_{\mathcal{B}} = W_{ei}$. Suppose now that preferences satisfy the classical assumptions and that the technology is convex. Then, for a natural subfamily of the linear family of choice sets, if the associated equal-opportunity–equivalence solution is to have an empty intersection with the no-domination solution, then the family should be a subsolution of the equal-income Walrasian family, as noted earlier. The resulting solution is the equal-income Walrasian solution. Here are other examples:

Definition Let $p \in \Delta^{\ell-1}$ be fixed. Given $M \in \mathbb{R}_+$, let $B^p(M) \equiv \{z \in \mathbb{R}^\ell_+ : pz \leq M\}$. Let $\mathcal{B}^p \equiv \{B^p(M) : M \in \mathbb{R}_+\}$.

Definition Given $p \in \Delta^{\ell-1}$, let $L(p) \equiv \{z \in \mathbb{R}^\ell_+ : pz = p\frac{\Omega}{|N|}\}$. Let $\mathcal{L} \equiv \{L(p) : p \in \Delta^{\ell-1}\}$.

Let $p \in \Delta^{\ell-1}$ be fixed. If the family $\mathcal{B}^p$ is used, and when efficiency is imposed, we obtain the solution that selects for each economy the efficient allocations that are Pareto-indifferent to $|N|$-lists of bundles whose values at the prices p (which have nothing to do with the prices of support of z) are equal. For the family $\mathcal{L}$, we obtain any efficient allocation such that each agent finds his component of it indifferent to the best bundle he could achieve if endowed with $\frac{\Omega}{|N|}$ and given access to a constant-returns-to-scale technology, the same for all agents, (Mas-Colell, 1980a). We describe below a characterization of the solution just defined on the basis of a monotonicity requirement (Section 7; Moulin, 1987a; Roemer and Silvestre, 1993, also explore the criterion).

Yet other examples of solutions can be obtained by having all agents face a hypothetical technology obtained from the actual one by imagining the productivity of one specific factor of production (alternatively of some subset of the factors of production) to be multiplied by some number, or by introducing a fixed cost of some factor of production (alternatively, introducing a fixed cost proportional to some fixed vector). Radial expansions and contractions of the production set can also be considered. A special case is the ratio solution (Kaneko, 1977a,b). A general application of the concept is due to Nicolò and Perea (2005). It covers private good allocations, cost sharing, and location of a public good.

6.3 No envy of opportunities

Given $e \equiv (R, \Omega, Y) \in \mathcal{E}^N_{pro}$ and $z \in P(e)$, let us now define the (implicit) opportunity set of agent $i \in N$ at z as the set of bundles whose value at the prices supporting z is no greater than the value of z_i (Varian, 1976). Equal opportunities, according to this definition, implies that $z \in W_{ei}(e)$. More generally, define the (once again, implicit) opportunity set of agent $i \in N$ at any $z \in Z(e)$ as the set of bundles whose value at the prices supporting agent i's upper contour set at z_i is no greater than the value of z_i (Archibald and Donaldson, 1979). For this definition to make sense, we should assume that these supporting prices are unique. Now, if the opportunity sets of two agents are not the same, they may not be related by inclusion, in contrast to the situation considered by Varian. In a production economy with differently productive agents, supporting prices need not be the same, so implicit opportunities cannot be equalized. One may require instead that no agent should prefer the implicit opportunities of another. To generalize this idea, let $\mathcal{B}$ be a family of choice sets:

Definition (Thomson, 1994a) Given $e \equiv (R, \Omega, Y) \in \mathcal{E}^N_{pro}$, the allocation $\boldsymbol{z \in Z(e)}$ **exhibits no envy of opportunities relative to** $\boldsymbol{\mathcal{B}}$ **for** $\boldsymbol{e}$, written as $\boldsymbol{z \in OF_{\mathcal{B}}(e)}$, if for each $i \in N$, there is $B_i \in \mathcal{B}$ such that $z_i \in B_i$ and z_i maximizes R_i on $\bigcup B_j$.

It is easy to check that if $\mathcal{B}$ is the family $\mathcal{L}$ of linear choice sets, then the resulting solution $OF_{\mathcal{B}}$ coincides with the equal-income Walrasian solution. Also, if $\mathcal{B}$ is the family of $|N|$-lists of bundles, we obtain the following concept, which generalizes both no-envy and egalitarian-equivalence:

Definition (Pazner, 1977) Given $e \equiv (R, \Omega, Y) \in \mathcal{E}^N_{pro}$, the allocation $\boldsymbol{z \in Z(e)}$ **is envy-free–equivalent for** $\boldsymbol{e}$ if there is $z' \equiv (z'_i)_{i \in N} \in \mathbb{R}^{\ell N}_+$ such that for each pair $\{i, j\} \subseteq N$, $z'_i \; R_i \; z'_j$, and $z \; I \; z'$. (Note that $z' \in Z(e)$ is not required.)

A model in which agents have preferences over commodity bundles, preferences over opportunity sets, and preferences over pairs of a commodity bundle and an opportunity set is studied by Tadenuma and Xu (2001). They say that a pair of a profile of an allocation and a profile of opportunity sets is **decentralizable for a profile of preference relations over bundles** if for each agent, his component of the allocation maximizes his preference relation over bundles in his component of the opportunity profile. They relate notions of no-envy according to these different relations, and offer characterizations of the Walrasian solution on the basis of decentralizability, and various forms of independence and no-envy requirements.

7 Monotonicity

For most of the analysis of the preceding sections, we have assumed the social endowment, and the production set when there was one, to be fixed. Here, we imagine changes in these data and we study how solutions respond to such changes. We start with the distribution of an unproduced social endowment. If it increases, we require that all agents should end up at least as well off as they were initially. We ask whether this requirement is compatible with our earlier equity criteria. As we will see, the answer depends on which of them is chosen. In economies with production, an appealing requirement is that if the technology improves, all agents should end up at least as well off as they were initially. We also consider a monotonicity requirement pertaining to economies with fixed resources but a variable number of agents. We would like everyone initially present to help supporting newcomers.

Each of the parameters just listed whose possible variations we consider belongs to a space equipped with an order structure, and the rule is required to respond well to changes that can be evaluated in that order. Under such assumptions, the change can be unambiguously evaluated in terms of welfares. More generally, we could consider simply replacing the initial value taken by the data with another value, and only require that the welfares of all agents should be affected in the same direction, namely that all should end up at least as well off as they were initially or that they all should end up at most as well off. We close with an application of the idea to situations when it is the replacement of the preferences of some agents that has to be faced, thereby obtaining an application of what we call later the "replacement principle".

7.1 Resource-monotonicity

Our first monotonicity property pertains to variations in resources. (For a survey of the various applications of the idea of monotonicity with respect to resources or opportunities, see Thomson, 1999c.) Given a class $\mathcal{R}$ of possible preferences over $\mathbb{R}^\ell_+$, an economy is a pair $(R, \Omega) \in \mathcal{E}^N \equiv \mathcal{R}^N \times \mathbb{R}^\ell_+$ and a solution is a mapping defined over $\mathcal{E}^N$ and taking its values in $\mathbb{R}^{\ell N}_+$.

Our requirement is that if the social endowment increases, all agents should end up at least as well off as they were initially. It allows for solution correspondences but given the choice of quantifiers, it implies *essential single-valuedness.*

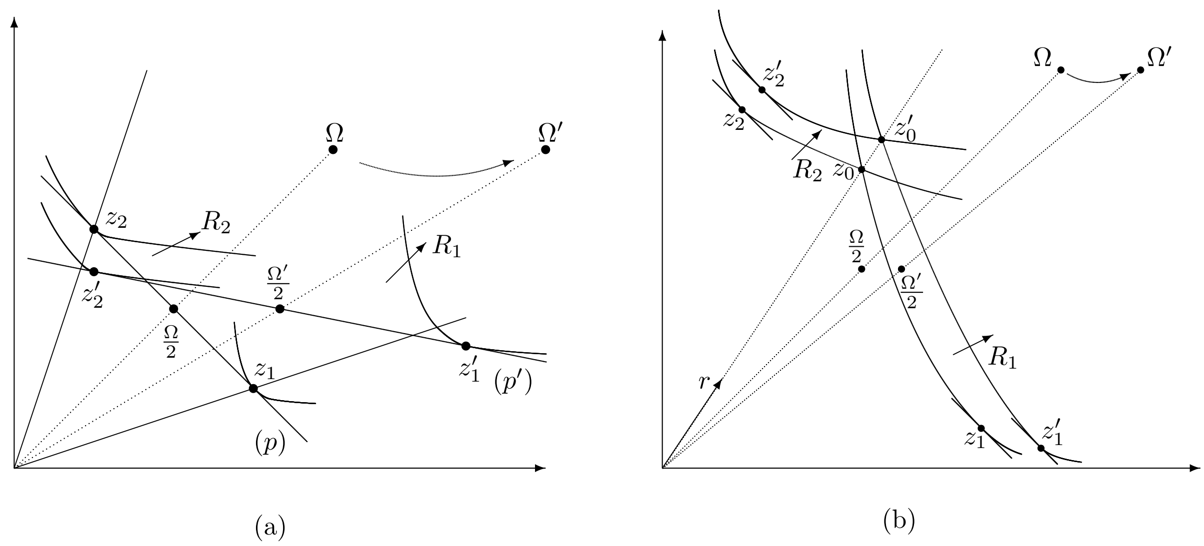


Figure 4: Two solutions, one resource-monotonic, the other not. (a) The equal-division Walrasian solution is not *resource-monotonic*. When the social endowment is Ω, the solution selects z. After the social endowment increases to Ω', it selects z', to which agent 1 prefers z. (b) The r-egalitarian-equivalence and Pareto solution is *resource-monotonic*. As the social endowment increases from Ω to Ω', the reference bundle associated with the allocation chosen by this solution can only move further out on the ray defined by r (it goes from z_0 to z'_0). This implies that both agents end up at least as well off as they were initially.

Resource-monotonicity: (Thomson, 1978; Roemer, 1986a,b; Chun and Thomson, 1988) For each $(R, \Omega) \in \mathcal{E}^N$, each $z \in \varphi(R, \Omega)$, each $\Omega' \in \mathbb{R}^{\ell}_{+}$, and each $z' \in \varphi(R, \Omega')$, if $\Omega' \geqq \Omega$, then $z' \; R \; z$.

It is easy to see that the equal-division Walrasian solution violates the property (Figure 4). This is so even on domains on which it is *essentially single-valued*, such as the domain of homothetic preferences (*essential single-valuedness* follows from the fact that in addition, endowments are equal), so writing the definition with existential instead of universal quantifiers for z and z' would not help. Our main result here is that the Walrasian solution is far from being the only one to suffer from this problem. To present it, we need to formally introduce as a solution the correspondence that associates with each economy the set of allocations at which no agent receives at least as much of each good as some other agent and more of at least one good. Let $D: \mathcal{E}^N \to \mathbb{R}^{\ell N}_{+}$ be this **no-domination** solution: $D(R, \Omega) \equiv \{z \in Z(R, \Omega)\colon$ for no pair $\{i, j\} \subseteq N, z_i \geq z_j\}$ (note that the definition does not involve preferences). This is a "large" correspondence. However, even under strong assumptions on preferences, it is not large enough to admit efficient *resource-monotonic* selections.

Theorem 7.1 (Moulin and Thomson, 1988) Domain: private goods; strictly monotonic, convex, and homothetic preferences. *No selection from the no-domination and Pareto solution is resource-monotonic.*

Theorem 7.1 is rather disappointing, but unfortunately, the situation is worse. Indeed, given $\varepsilon \geq 0$, let us replace no-domination by the requirement that each agent should be made at least as well off as he would be by consuming ε percent of the social endowment (Moulin and Thomson, 1988). The smaller ε is, the weaker the requirement. However, no matter how small ε is, provided it is positive, the incompatibility persists. Alternatively, and here too, for each $\varepsilon > 0$, no-domination could be replaced in this theorem by the requirement that no agent should receive less than ε percent of what each other agent receives.

It is only at the limit, when $\varepsilon = 0$, that is, when no restriction is placed on the extent to which an agent might be discriminated against, that a positive result is obtained. Indeed, all of the egalitarian-type solutions defined as follows are *resource-monotonic*: for each agent, choose a continuous numerical representation of his preferences—a welfare index—and then the allocation(s) whose image in welfare space is maximal among all feasible points at which the welfare gains from the image of the allocation consisting entirely of zero bundles are equal. Any such solutions is *resource-monotonic* because an increase in the social endowment causes the feasible set to expand. It then becomes possible to move further up along the equal-gains line.[37] More generally, each equal-opportunity–equivalent solution with respect to a family of choice sets satisfying minimal regularity properties is *resource-monotonic.*

[37]It is on this fact that Kalai's (1977) well-known characterization of the egalitarian solution to the bargaining problem is based. In economies in which agents are individually endowed, this fact also underlies the proof of existence of rules that are "own-endowment monotonic", that is, such that when an agent's endowment increases, he ends up at least as well off as he was initially. Aumann and Peleg (1974) provide an example of an economy with continuous, strictly monotonic, convex, and homothetic preferences revealing that this rule violates the property. In the example, the Walrasian allocation is in fact unique for each endowment profile. The strategic implication of such a violation is that an agent may benefit from destroying part of his endowment (Hurwicz, 1978). The Aumann-Peleg example can also be used to show that the Walrasian solution is such that as one agent transfers some of his endowment to some other agent, both may benefit (Gale, 1974), a phenomenon related to the "transfer problem", well-known to international trade theorists.

One can even go beyond this class without losing *resource-monotonicity*: each solution defined by selecting a maximal point of equal welfares, using as welfare index for each agent a function that takes value zero at the zero bundle, will do.

A result related to Theorem 7.1 is that no selection from the Pareto solution satisfies *Property* α (Subsection 4.1) and *resource-monotonicity* (Maniquet and Sprumont, 2000).

A requirement in the spirit of, but weaker than, *resource-monotonicity*, is that when the social endowment increases, the bundle assigned to no agent should be dominated, good by good, by the bundle assigned to him initially. No selection from the no-envy and Pareto solution satisfies it (Geanakoplos and Nalebuff, 1988).[38] Also, no matter how small $\varepsilon > 0$ is, no selection from the Pareto solution that always makes each agent at least as well off as he would be by consuming ε percent of the social endowment satisfies it (Moulin, 1991).

The proofs of the negative results just described rely on the admissibility of preferences with indifference curves that are close to right angles. In their "ε variants", the smaller ε is, the closer to right angles indifference curves are. Do the results persist if preferences are required to satisfy some minimal degree of substitutability? The answer is no. For instance, it follows from Polterovich and Spivak (1980, 1983) that when preferences satisfy gross substitutability and all goods are normal, the equal-division Walrasian solution is *resource-monotonic* (Moulin and Thomson, 1988).

Let $r \in \Delta^{\ell-1}\backslash\{0\}$ be given. By using as each agent i's welfare index the function $t\colon \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ defined by $z_i \ I_i \ t(z_i)r$, we deduce from an earlier observation that the r-egalitarian-equivalence and Pareto solution is *resource-monotonic*. (In fact, the rule is such that the welfares of all agents are affected in the same direction by any variation in the social endowment, whether or not the values it takes are related by domination.) This property of the r-egalitarian-equivalence family of solutions is a very strong point in their favor. In the pages to follow, we will encounter a number of additional arguments

[38] Geanakoplos and Nalebuff state their result for correspondences, requiring that when the social endowment increases, then for each agent, there should be at least one good, one allocation in the initial economy and one allocation in the final economy at which he receives more of that good after the enlargement. This choice of quantifiers makes the monotonicity requirement weaker. Their non-existence proof involves economies with more than two agents.

of that nature lending them additional support. What is critical is that the reference bundle be independent of the social endowment. We saw that requiring that it be proportional to the social endowment guarantees that the equal-division lower bound is met. It is an implication of Theorem 7.1 that if efficiency is imposed, this bound is obtained at the price of *resource-monotonicity*. (Without efficiency, the rule that always selects equal division would of course be acceptable.)

For the quasi-linear case, we have good news. Given an economy e in which preferences can be represented by functions that are quasi-linear with respect to a particular good (the same for all agents), and using such representations as welfare indices—the special good is used as an "accounting good" then—consider the coalitional game $w(e)$ defined as follows: set the worth of each coalition to be the maximal aggregate welfare it can reach if given access to the entire social endowment, its "free-access" aggregate welfare, in the terminology of Section 5. The next theorem describes additional properties of preferences under which the Shapley value (Shapley, 1953) applied to the coalitional game $w(e)$ induces a *resource-monotonic* solution. First, say that **two goods j and k are substitutes for a function $f\colon \mathbb{R}^{\ell} \to \mathbb{R}$** if the amount by which it increases as its k-th argument increases by some arbitrary amount is a decreasing function of its j-th argument: for each $x_i \in \mathbb{R}^{\ell}_{+}$ and each pair $\{a, b\} \subset \mathbb{R}_{+}$, we have $f(y_i + be^k) - f(y_i) \geq f(y_i + ae^j + be^k) - f(y_i + ae^j)$, where e^j denotes the j-th unit vector. Also, **the function f satisfies substitutability** if any two goods are substitutes for f. (Writing the condition when $j = k$ is equivalent to saying that f is concave in y_j.) Finally, **the coalitional game $w(e)$ satisfies substitutability** if each of its coordinates does. Now, we have:

Proposition 7.1 (Moulin, 1992b) Domain: private goods; quasi-linear preference profiles; free-access coalitional game associated to each economy satisfies substitutability. *The Shapley value, when applied to these games, induces a resource-monotonic solution.*

If there is only one good in addition to the accounting good, substitutability of v_i is equivalent to its concavity. Another application of Proposition 7.1 is when there are only two goods in addition to the accounting good, and for each $i \in N$, the function v_i is concave and submodular over $\mathbb{R}^2_{+}$. It also applies in the case of many goods if for each $i \in N$, the function v_i is twice continuously differentiable in the interior of $\mathbb{R}^{\ell}_{+}$, strictly concave, exhibits gross substitutability, and the marginal utility of each good at 0 is infinite.

Studies of monotonicity in abstract settings are due to Moulin (1989, 1990d).

7.2 Welfare-domination under preference-replacement

In the statement of the monotonicity properties discussed above, we focused on one of the parameters defining the economy, and considered changes in that parameter that could be unambiguously described as good for a certain group of agents; we required that as a result of the change, all agents should be made at least as well off as they were initially. We could in fact imagine *arbitrary* changes in the parameter and demand that the welfares of all relevant agents should be affected in the same direction: as a result of the replacement, they should all be made at least as well off as they were initially or they should all be made at most as well off. This is the most general way of expressing the idea of solidarity among agents. It is referred to in Thomson (1990c, 1997) as the **replacement principle**. The principle can also be applied to changes in parameters taken from spaces that are not equipped with order structures. A primary example of such a parameter are the preferences of an agent. This consideration leads us to the following requirement. (The literature devoted to its analysis in various models is surveyed in Thomson, 1999a.)

Welfare-domination under preference-replacement: For each $(R, \Omega) \in \mathcal{E}^N$, each $z \in \varphi(R, \Omega)$, each $i \in N$, each $R'_i \in \mathcal{R}$, and each $z' \in \varphi(R'_i, R_{-i}, \Omega)$, either $z'_{N\setminus\{i\}}\ R_{N\setminus\{i\}}\ z_{N\setminus\{i\}}$ or $z_{N\setminus\{i\}}\ R_{N\setminus\{i\}}\ z'_{N\setminus\{i\}}$.

It is obvious that the equal-division Walrasian solution violates the property, even in the two-good case. In general, a change in some agent's preferences is accompanied by a change in the equilibrium prices, and if at least two of the other agents are initially on opposite sides of the market, any such change will make one of them better off and the other worse off. Unfortunately, this difficulty is widely shared:

Theorem 7.2 (Thomson, 1996) Domain: private goods; strictly monotonic and homothetic preferences. *No selection from the no-envy and Pareto solution satisfies welfare-domination under preference-replacement.*

The proof is by means of two-good and three-agent economies. In fact, even if no-envy is weakened to no-domination, an incompatibility with efficiency holds (Kim, 2001). It is clear however that selections from the egalitarian-equivalence and Pareto solution exist that satisfy *welfare-domination under preference-replacement*. Simply choose a continuous, unbounded, and monotone path in commodity space and require the reference bundle to belong to the path. If the path goes through equal division, the resulting rule is also a selection from the equal-division lower bound solution. These desirable properties hold whether or not the path is a ray.

The replacement principle can be applied to the joint replacement of resources and preferences. A general result describing its implications then is given by Sprumont (1996). His formulation covers as special cases the classical model but also public good models. Suppose that it is meaningful to compare the relative treatment of agents i and j, when agent i has preferences R_i and is assigned a consumption on a certain indifference curve of R_i and agent j has preferences R_j and is assigned a consumption on a certain indifference curve of R_j, and that in fact a social order exists on the space of all such pairs. The result is that under a richness property of its range, and if there are at least three agents, a rule satisfies what can be called *welfare-domination under resource-and-preference replacement* if and only if there is an ordering of the kind just described such that for each problem, the rule selects an allocation at which all agents are assigned consumptions so that they are all treated equally well according to this ordering.

7.3 Technology-monotonicity

Another interesting comparison can be made between two economies that differ only in their technologies. Suppose that the technology of one dominates the technology of the other. It seems natural to require that in the first one, all agents should be made at least as well off as they are in the second one. In order to formally state the property, we need to reintroduce production possibilities in the notation. A technology is a subset Y of commodity space $\mathbb{R}^\ell$. Let $\mathcal{Y}$ be a class of admissible technologies. Here, an economy is a triple $(R, \Omega, Y) \in \mathcal{R}^N \times \mathbb{R}^\ell_+ \times \mathcal{Y}$. Let $\mathcal{E}^N_{pro}$ be our generic notation for a domain of economies.

Technology-monotonicity: (Roemer, 1986) For each $(R, \Omega, Y) \in \mathcal{E}^N_{pro}$, each $Y' \in \mathcal{Y}$, each $z \in \varphi(R, \Omega, Y)$, and each $z' \in \varphi(R, \Omega, Y')$, if $Y' \supseteq Y$, then

$z' \; R \; z$.

The requirement is satisfied by certain selections from the egalitarian-equivalence and Pareto solution. In the case of two goods, a characterization of a particular one is obtained by imposing it together with a few other minimal requirements, as explained next.

Suppose first that good 1 is used to produce good 2 according to a nowhere-decreasing-returns-to-scale technology Y. Given a group N of agents with preferences defined on $\mathbb{R}^2_+$, given some social endowment Ω of good 1, which can be consumed as such or used as input in the production of good 2, and given Y, let $\varphi(R, \Omega, Y)$ be the set of allocations selected by the solution φ. Here, we denote by $\underline{B}_{ed+Y}(R, \Omega, Y)$ the set of allocations such that each agent finds his bundle at least as desirable as the best bundle he could achieve if endowed with $\frac{\Omega}{|N|}$ and given unhampered access to the technology Y: the **equal-division free-access lower bound solution**.

The next solution illustrates the definition of equal-opportunity–equivalence of Subsection 6.2. The definition is depicted in Figure 5 for a nowhere-increasing-returns-to-scale technology.

Definition (Mas-Colell, 1980a) Given $e \equiv (R, \Omega, Y) \in \mathcal{E}^N_{pro}$, the allocation $\boldsymbol{z} \in \boldsymbol{P(e)}$ **is a constant-returns-to-scale–equivalent allocation for** $\boldsymbol{e}$, written as $\boldsymbol{z} \in \boldsymbol{CRS^{\simeq}(e)}$, if there is a constant returns-to-scale technology such that for each $i \in N$, $z_i \; I_i \; z_i^*$, where z_i^* maximizes agent i's preferences if given access to $(\frac{\Omega}{|N|}, Y)$.

Theorem 7.3 (Moulin, 1987b; 1990d) Domain: one-input and one-output production economies; social endowment of the input; preferences are strictly monotonic with respect to the input and monotonic with respect to the output; (a) nowhere-decreasing-returns-to-scale technologies; (b) nowhere-increasing-returns-to-scale technologies (alternatively, convex technologies). (a) *The only selection from the equal-division free-access lower bound and Pareto solution satisfying* Pareto-indifference *and* technology-monotonicity *is the constant-returns-to-scale–equivalence solution.* (b) *A parallel statement holds for selections from the equal-division free-access upper bound.*

Although in Part (a) of the theorem, the bounds on welfares are individual bounds, the solution that emerges happens to satisfy the requirement that no group of agents should be able to make each of its members at least as well

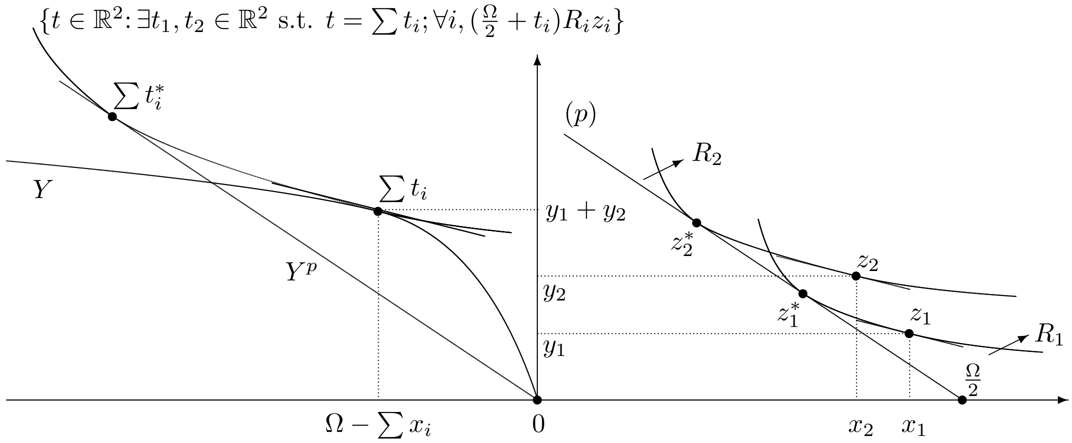


Figure 5: Constant-returns-to-scale equivalence solution. This solution selects the unique (up to Pareto-indifference) efficient allocation such that, for some reference constant-returns-to-scale technology Y^p, each agent $i \in N$ is indifferent between his bundle z_i and the best bundle he could reach if endowed with an equal share of the social endowment of the input and given access to that reference technology.

off, and at least one of them better off, if each of its members is endowed with an equal share of the social endowment and the group is given unhampered access to the technology. A parallel statement holds for Part (b).

One could be more demanding and consider simultaneous changes in several of the parameters describing the problem. For instance, suppose that resources and technologies both change. This may or may not lead to an enlargement of opportunities, but Dutta and Vohra (1993), who study this possibility, require of a correspondence that if an enlargement of the set of feasible profiles of welfare levels does occur, each allocation chosen initially should be welfare dominated by some allocation chosen after the change, and that each allocation chosen after the change should welfare dominate some allocation chosen initially. Let us refer to this requirement as **opportunity-monotonicity**. They also require, under the name of ***r*-equity**, that in an exchange economy in which there is only some amount of good r to divide, equal division should be chosen. They consider an invariance requirement that also depends on the choice of a good, say r, so we call it ***r*-invariance**. It is somewhat technical and not motivated by normative considerations, so we do not state it explicitly, only noting that it is a weak version of the invariance requirement shown by Maskin (1999) to be critical to the possibility of implementation. This requirement is usually called "Maskin monotonicity", but we will use the more descriptive expression of **invariance under monotonic transformations of preferences**: if an allocation is chosen for some economy and preferences change in such a way that the restriction of each agent's new lower contour set at his component of the allocation to the set of feasible allocations contains the corresponding set for his initial preferences, then the allocation should still be chosen for the new economy:

Theorem 7.4 (Dutta and Vohra, 1993) Domain: private goods; monotonic and convex preferences such that each indifference curve crosses the r-th axis; production set is closed, contains the origin, and exhibits free disposal (non-convex sets are allowed; exchange economies are included); set of feasible welfare profiles (using arbitrary continuous numerical representations of preferences), is bounded. *Up to Pareto-indifference, (a) the r-egalitarian equivalence and Pareto solution is the only subsolution of the Pareto solution satisfying r-equity and opportunity-monotonicity; (b) on the subdomain of exchange economies, it is the only subsolution of the Pareto solution satisfying r-equity, r-invariance, and opportunity-monotonicity.*

Another independence condition is **contraction-independence**: if an allocation is chosen for some economy, the technology contracts but the allocation remains feasible, then it should still be chosen.

Several characterizations are available that involve this requirement. For the first one, we need an addition definition, due to Roemer and Silvestre (1993). They consider economies with arbitrarily many goods and identify general conditions under which existence is guaranteed. We limit attention to economies with two goods:

Definition Given $e \equiv (R, \Omega, Y) \in \mathcal{E}^N_{pro}$, a class of production economies with two goods, one of them being used as an input in the production of the other, the allocation $\boldsymbol{z} \in \boldsymbol{P(e)}$ **is a proportional allocation for** $\boldsymbol{e}$, written as $\boldsymbol{z} \in \boldsymbol{Pro(e)}$, if either $z = 0$ or for each pair $\{i, j\} \subseteq N$, the ratio of agent i's input contribution over his output consumption is equal to the corresponding ratio for agent j.

Theorem 7.5 (Moulin, 1990d) Domain: one-input one-output production economies; monotonic and strictly convex preferences; convex technologies. *(a) If a selection from the identical-preferences lower bound satisfies contraction-independence, then it contains the equal-income Walrasian solution. (b) The proportional rule is the only selection from the Pareto and* *free-access upper bound solution satisfying* contraction-independence *and* invariance under monotonic transformations of preferences.

We also have a partial characterization based on the constant-returns-to-scale lower bound:

Theorem 7.6 (Maniquet, 1996b) Domain: one-input one-output production economies; monotonic and convex preferences; nowhere-increasing-returns-to-scale technology. *If a subsolution of the constant-returns-to-scale lower bound solution satisfies* Pareto indifference *and* contraction-independence, *then it contains the constant-returns-to-scale–equivalence solution.*

The next theorem involves **weak invariance under monotonic transformations of preferences**, obtained from the requirement defined above by using the hypothesis of inclusion of upper contour sets instead of inclusion of restricted upper contour sets (Gevers, 1986).

Theorem 7.7 (Maniquet, 2002) Domain: one-input one-output production economies; monotonic and convex preferences; convex technologies. *The proportional rule is the only subsolution of the Pareto solution that selects the efficient and proportional allocations when they exist and satisfies* weak invariance under monotonic transformations of preferences.

We close this discussion of production economies with a mention of two other interesting monotonicity requirements. The first one pertains to situations where agents are differentiated by their input contributions. It simply says that if the contribution of an agent increases, he should end up at least as well off as he was initially.

The second one pertains to situations in which agents differ in their productivities. It states the corresponding requirement that if an agent's productivity increases, then again, he should end up at least as well off as he was initially.

Technology-monotonicity is also considered Moulin and Roemer (1989). They focus on economies with two agents equipped with utility functions, the same for both, but whose productivities may differ.

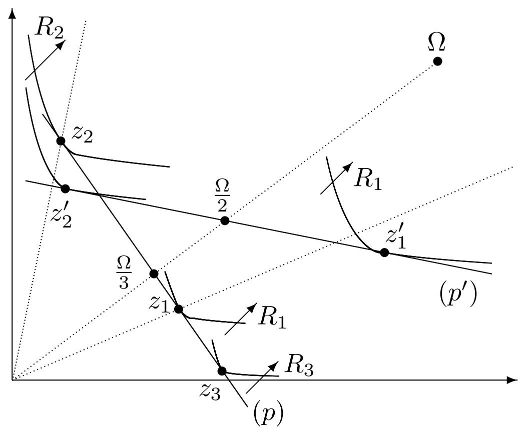


Figure 6: The equal-division Walrasian solution is not population-monotonic. In the example depicted here, when all three agents are present, the solution selects z. If agent 3 is not present, it selects z', to which agent 2 prefers z.

7.4 Population-monotonicity

Next, and returning to the problem of distributing a fixed bundle of goods, we consider a monotonicity property that pertains to situations in which resources are fixed but the population of agents varies. If it enlarges, we require that all agents initially present should end up at most as well off as they were initially. (The literature devoted to the analysis of this requirement in various models is surveyed in Thomson, 1995b). For a formal statement, recall our notation for solutions that accommodate variable populations (Section 2).

Population-monotonicity: (Thomson, 1983b) For each $N \in \mathcal{N}$, each $(R, \Omega) \in \mathcal{E}^N$, each $z \in \varphi(R, \Omega)$, each $N' \subseteq N$, and each $z' \in \varphi(R_{N'}, \Omega)$, we have $z' \; R_{N'} \; z_{N'}$.

It is easy to see that the equal-division Walrasian solution violates this property (Figure 6). In fact, it may violate it in economies with monotonic, convex, and homothetic (or quasi-linear) preferences (Chichilnisky and Thomson, 1987). More seriously, we have the following general impossibility:[39]

Theorem 7.8 (Kim, 2004) Domain: private goods; strictly monotonic, convex, and homothetic preferences. *No selection from the no-envy and Pareto solution is population-monotonic.*

A related result holds for economies with a large number of agents modelled as a continuum (Moulin, 1990c, 1991). It is based on the following facts:

[39]The proof requires that there be at least eight agents.

(i) in such an economy, and if preferences are sufficiently diverse, any envy-free allocation is an equal-division Walrasian allocation (Section 4.6) and (ii) in economies with finitely many agents, the equal-division Walrasian solution is not *population-monotonic* (Figure 6). The proof consists in approximating a finite pair of economies illustrating (ii) by a continuum economy satisfying (i).

Theorem 7.8 can be strengthened in the same manner as Theorem 7.1 was: on the same domain, for each $\varepsilon > 0$, no selection from the ε—no-domination and Pareto solution is *population-monotonic* (Kim, 2004) (the smaller ε is, the larger the number of agents required for the proof). Just as was the case for Theorem 7.1 and its "ε variant", Theorem 7.8 and its "ε variant" rely on the admissibility of preferences whose indifference curves can be arbitrarily close to right angles. Positive results hold, at least for the equal-division Walrasian solution, if preferences satisfy some minimal degree of substitutability. In particular, it follows from Polterovich and Spivak (1980, 1983) that this solution is *population-monotonic* if preferences exhibit the gross-substitutability property and all goods are normal (Fleurbaey, 1995c).

There is a connection between *population-monotonicity* being violated by the equal-division Walrasian rule to its being subject to the "transfer paradox": in an economy with individual endowments, a transfer of endowment from an agent to some other agent may make the donor better off and the recipient worse off. The following results pertain to homothetic preferences: In the absence of substitution effects, the rule is subject to the transfer paradox if and only if it violates *population-monotonicity.* In the presence of substitution effects, it may violate *population-monotonicity* even in situations where no transfer paradox can occur (Jones, 1987).

What if other distributional requirements are imposed? If preferences are strictly monotonic, the Ω-egalitarian-equivalence and Pareto solution is *population monotonic* (Thomson, 1987). Recall that this solution also meets the equal-division lower bound. More generally, and still if preferences are strictly monotonic, the equal-opportunity–equivalence and Pareto solutions of Subsection 6.2 are *population-monotonic.*

For quasi-linear preference profiles, we have a counterpart of Proposition 7.1:

Proposition 7.2 (Moulin, 1992b) Domain: private goods with a positive amount of money; quasi-linear preferences; free-access coalitional game associated with each economy satisfies substitutability. *The Shapley value, when applied to these games, induces a population-monotonic solution.*

Selections from the equal-division lower bound and Pareto solution exist that are *population-monotonic*. A constructive algorithm producing such a solution can indeed be defined (Moulin, 1990b), but in contrast with the solution induced by the Shapley value, this solution is not *resource-monotonic*.

If we apply the replacement principle of Subsection 7.2 to the *joint* replacement of preferences and population, then under the additional requirement of *replication-invariance*, only one solution emerges, namely the selection from the egalitarian-equivalence solution for which the reference bundle is proportional to the social endowment.

Theorem 7.9 (Sprumont and Zhou, 1999) Domain: private goods; strictly monotonic and convex preferences. *The Ω-egalitarian-equivalence and Pareto solution is the only selection from the equal-division lower bound and Pareto solution satisfying* replication-invariance *and* welfare-domination under preference-and-population-replacement.

Interestingly, this uniqueness result fails if the joint *welfare-domination* requirement is replaced by two separate requirements, one pertaining to changes in preferences and the other pertaining to changes in population.

In a model with infinitely many agents modelled as a continuum, versions of Theorem 7.9 hold that do not involve *replication-invariance* (Sprumont and Zhou, 1999).

7.5 Monotonicity in economies with individual endowments

If the issue is that of allocating gains from trade, other appealing monotonicity requirements can be imposed. One is that if an agent's endowment increases, he should be made at least as well off as he was initially. Another is that under the same hypotheses, nobody else should be made worse off than he was initially. Here are the formal definitions:

Own-endowment monotonicity: For each $(R, \omega) \in \mathcal{R}^N \times \mathbb{R}_+^{\ell N}$, each $z \in \varphi(R, \omega)$, each $i \in N$, each $\omega'_i \in \mathbb{R}_+^{\ell}$, and each $z' \in \varphi(R, \omega'_i, \omega_{-i})$, if $\omega'_i \geq \omega_i$, then $z'_i \; R_i \; z_i$.

No negative effects on others: Under the hypotheses of the previous definition, $z'_{N\setminus\{i\}} \; R_{N\setminus\{i\}} \; z_{N\setminus\{i\}}$.

It is easy to define selections from the individual-endowments lower-bound and Pareto solution that are *own-endowment monotonic*. However, we also have impossibilities:

Theorem 7.10 (Thomson, 1987a) Domain: private goods; strictly monotonic, convex, and homothetic preferences; individual endowments. (a) *No selection from the no-envy in trades and Pareto solution satisfies either* own-endowment monotonicity *or* no negative effect on others.[40] (b) *No selection from the egalitarian-equivalence and Pareto solution satisfies* no negative effect on others.[41]

When population varies, the appropriate form of the idea of *population-monotonicity* is that the welfares of all agents who are present before and after the change should be affected in the same direction. It is easy to see that the Walrasian solution violates the property, even when preferences are homothetic and endowments proportional (assumptions that guarantee its *single-valuedness*).

However, the selections from the egalitarian-equivalence in trades and Pareto solution, obtained by requiring the reference trade to lie on a monotone path satisfy the requirement (Thomson, 1995a). They also meet the individual-endowments lower bound.

8 Consistency and related properties

Here, we return to situations in which both the population of agents and the resources available may vary, but this time, our focus is on a variety of invariance properties. These properties can be interpreted as formalizing tradeoffs between equity and efficiency objectives with objectives of informational simplicity. Unless otherwise indicated, this section is based on Thomson (1988). (The literature devoted to the analysis of the properties in various models is surveyed in Thomson, 1990b, 1995d.)

[40] The first part of this statement is an implication of the fact that on this domain no selection from the no-envy in trades and Pareto solution is immune to manipulation through withholding of endowments (Postlewaite, 1978).

[41] An ε variant of this result holds, analogous to the ε variant of Theorem 7.1.

8.1 Consistency and converse consistency

We have already encountered *replication-invariance* (Subsection 4.6). A converse of this requirement is that if an allocation that is chosen for a replica economy happens to be a replica allocation (of the same order), then the model allocation should be chosen for the model economy:

Division-invariance: For each $N \in \mathcal{N}$, each $(R, \Omega) \in \mathcal{E}^N$, each $z \in \varphi(R, \Omega)$, each $N' \subset N$, each $(R', \Omega') \in \mathcal{E}^{N'}$, and each $k \in \mathbb{N}$, if (R, Ω) is a k-replica of (R', Ω') and z is the corresponding k-replica of some $z' \in Z(R', \Omega')$, then $z' \in \varphi(R', \Omega')$.

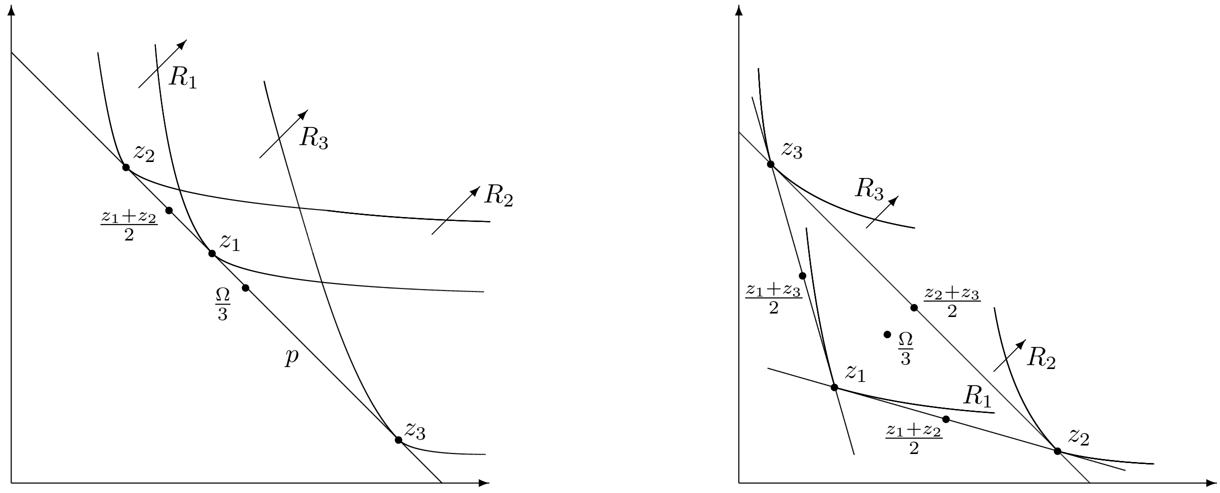
Given a group $N \in \mathcal{N}$, and an allocation z chosen for some economy $(R, \Omega) \in \mathcal{E}^N$, consider some subgroup $N' \subset N$, and the problem of allocating among its members the resources that it has received in total. Our next requirement, the central one in this section, is that the restriction of z to the subgroup should be chosen for this economy, $(R_{N'}, \sum_{N'} z_i)$, the **reduced economy of e with respect to N' and z**:

Consistency: For each $N \in \mathcal{N}$, each $(R, \Omega) \in \mathcal{E}^N$, each $z \in \varphi(R, \Omega)$, and each $N' \subset N$, we have $z_{N'} \in \varphi(R_{N'}, \sum_{N'} z_i)$.

A counterpart of this requirement when the population of agents enlarges is the following: given some allocation z that is feasible for some economy, check whether the restriction of z to each subgroup of two agents is chosen for the problem of allocating between them what they have received in total. If the answer is yes for each such subgroup, then one can say that each agent is in a sense treated fairly in relation to each other agent; then, we require that z itself should be chosen for the initial economy:

Converse consistency: For each $N \in \mathcal{N}$, each $e \equiv (R, \Omega) \in \mathcal{E}^N$, and each $z \in P(e)$, if for each $N' \subseteq N$ with $|N'| = 2$, $z_{N'} \in \varphi(R_{N'}, \sum_{N'} z_i)$, then $z \in \varphi(e)$.

An alternative formulation is obtained by writing the hypotheses for all proper subgroups of N, as opposed to all subgroups of two agents, but an easy induction argument shows that this apparently weaker property is equivalent to the one just stated.



(a) (b)

Figure 7: The equal-division Walrasian solution is consistent but not conversely consistent. (a) For the three-agent economy represented here, the solution selects the allocation z. For the economy with agent set $\{1, 2\}$ in which the amount to divide is $\Omega - z_3$, it selects (z_1, z_2) (the same prices can serve as equilibrium prices). (b) If kinks in indifference curves are permitted, the solution is not *conversely consistent*. We consider the efficient allocation z in the three-agent economy $e \equiv (R_1, R_2, R_3, \Omega)$. It is such that for each pair $\{i, j\}$, $(z_i, z_j) \in W_{ed}(R_i, R_j, z_i + z_j)$. Yet $z \notin W_{ed}(e)$.

Several results below require preferences to be **smooth**: at each $z_i \in \mathbb{R}^{\ell}_{++}$, agent i's upper contour set has a unique hyperplane of support. Let $\mathcal{E}^N_{sm}$ be a class of smooth economies.

It is easy to see that the Pareto solution is *consistent*. Under smoothness of preferences, it is also *conversely consistent*. (Goldman and Starr, 1982, establish conditions under which the hypothesis that no group of $t < |N|$ agents can achieve any gain from trade implies that the entire set of agents cannot either.) It is *replication-invariant* in general (convexity of preferences is important here). The no-envy solution satisfies all of the above properties, and the egalitarian-equivalence solution only fails *converse consistency*. This is also the case for the equal-division Walrasian solution. Its *consistency* is illustrated by Figure 7a. (The same prices remain equilibrium prices in a reduced economy but there could be other equilibrium allocations, some of which supported by prices other than the equilibrium prices of the initial economy.) Figure 7b shows that the equal-division Walrasian solution is not *conversely consistent*. However, if preferences are smooth and corners are excluded, the property does hold. The equal-division core satisfies *division-invariance* but none of the other properties.

We will complement the four properties by efficiency and fairness requirements, obtaining characterizations of the equal-division Walrasian solution. The first one is a direct consequence of the well-known fact that under replication, the core shrinks to the set of Walrasian allocations (Debreu and Scarf, 1963). Many variants of this theorem have been proved. It is stated in this form in Thomson (1988). Nagahisa (1994) also discusses the issue.

Theorem 8.1 Domain: private goods (strictly positive social endowment); locally non-satiated and convex preferences. *If a subsolution of the equal-division core is replication-invariant, it is a subsolution of the equal-division Walrasian solution.*

Theorem 8.2 (Varian, 1974) Domain: private goods; monotonic and convex preferences. *If a subsolution of the group no-envy solution is replication-invariant, it is a subsolution of the equal-division Walrasian solution.*

Theorem 8.3 (Thomson, 1988) Domain: private goods; monotonic, convex, and smooth preferences. *If a subsolution of the equal-division lower bound and Pareto solution is replication-invariant and consistent, it is a subsolution of the equal-division Walrasian solution.*

A characterization of the equal-division Walrasian solution on the basis of the concept of a strictly envy-free allocation (Subsection 4.4) due to Zhou (1992) is equivalent to Theorem 8.3. Zhou also offers an estimate of the speed of convergence to Walrasian allocations as the order of replication increases.

The issue of *consistency* in economies with a large number of agents modelled as a continuum has also been addressed. For this model, we adapt the definition of a strictly envy-free allocation as one such that the set of agents each of whom prefers the average bundle received by some group to his own bundle has measure zero. Under certain assumptions, a central one being smoothness of preferences, the equal-division Walrasian solution coincides with the strict no-envy and Pareto solution (Zhou, 1992). Also, it is the only subsolution of the equal-division lower bound and Pareto solution satisfying *consistency* (Thomson and Zhou, 1993). As compared to Theorem 8.3, it is noteworthy that this second result does not involve *replication-invariance*. Also, it extends to economies with possibly satiated preferences, yielding a characterization of the "equal-slack Walrasian solution" (Mas-Colell, 1992), which differs from the standard notion in that each agent's income is obtained by adding to the value of his endowment a supplementary income, the same for all agents. Finally, it holds for solutions defined on a domain consisting of a single economy and all of its possible reductions. The case of economies with both atoms and an atomless sector is considered by Zhou (1992). There, a Walrasian conclusion is reached for the members of the atomless sector only.

The following characterization of the equal-division Walrasian solution involves *converse consistency*:

Theorem 8.4 (Thomson, 1995d) Domain: private goods; monotonic, convex, and smooth preferences. (a) *If a subsolution of the equal-division lower bound and Pareto solution satisfies* anonymity *and* converse consistency, *then on the subdomain of two-agent economies, it is a subsolution of the equal-division Walrasian solution.* (b) *If in fact coincidence occurs on that subdomain, it is an arbitrary solution containing the equal-division Walrasian solution for all other cardinalities.*

The next result involves two new axioms. **Uniform treatment of uniforms** says that if all agents have the same preferences, only allocations consisting of bundles that are indifferent to each other according to these preferences should be chosen (Maniquet, 1996). **Juxtaposition-invariance** says that if an allocation is efficient for some economy and it happens to be obtained by juxtaposing two allocations that are chosen for two subeconomies with equal per-capita social endowments, then it should be chosen (Thomson, 1988):

Theorem 8.5 (Maniquet, 1996) Domain: private goods; monotonic, convex, and smooth preferences such that no positive bundle is indifferent to a bundle having at least one zero coordinate. *The equal-division Walrasian solution is the only subsolution of the Pareto solution satisfying* uniform treatment of uniforms, juxtaposition-invariance, *and* consistency.

The next results pertain to production economies. In formulating *consistency* for a production economy, the issue arises of how to adjust the technology to reflect the fact that agents leave with their consumptions. The simplest one is to translate the production set by the sum of the bundles taken with them by the agents who leave. Standard classes of technologies are not closed under this operation however, and adjustments have to be made to ensure that the "reduced" production set is admissible. Adjustments are also needed for *replication-invariance*.

A requirement related to *invariance under monotonic transformations of preferences* is that if an allocation is chosen for some economy and preferences change in such a way that for each agent, his indifference curve through his assigned bundle remains the same, then the allocation should still be chosen. Let us refer to it as **invariance under restricted monotonic transformations of preferences** (Maniquet, 2002).

Theorem 8.6 (Maniquet, 2002) Domain: one-input one-output production economies; monotonic and convex preferences; production set is closed under disposal. *If a subsolution of the Pareto and constant-returns-to-scale lower bound solution satisfies invariance under restricted monotonic transformations of preferences, replication-invariance, and consistency, then it is subsolution of the constant-returns-to-scale–equivalent solution.*

Next are characterizations of two *essentially single-valued* solutions, the **equal-wage–equivalent and Pareto solution**, which selects the allocations for which there is a reference wage such that each agent finds his bundle indifferent to the best bundle he could achieve by maximizing his preferences on a budget set defined by this wage rate. The **output-egalitarian-equivalence and Pareto solution**, selects the efficient allocations that each agent finds indifferent to a common consumption consisting of only some amount of the output. We will impose the self-explanatory notion of **equal welfares for equal preferences**.

Theorem 8.7 (Fleurbaey and Maniquet, 1999) Domain: private goods; preferences are strictly monotonic with respect to output and monotonic with respect to input, and convex; unrestricted technologies. *The equal-wage–equivalent and Pareto solution is the only essentially single-valued selection from the constant-returns-to-scale lower bound solution satisfying Pareto-indifference, equal welfares for equal preferences, contraction-independence, and consistency.*

Note the difference of domains in the next theorem:

Theorem 8.8 (Fleurbaey and Maniquet, 1999) Domains: private goods; preferences are strictly monotonic with respect to output and monotonic with respect to input, and convex; technologies are one of the following: (a) they are unrestricted, (b) they exhibit nowhere-increasing-returns-to-scale, (c) they are concave. *The output-egalitarian-equivalence and Pareto* *solution is the only essentially single-valued selection from the work-alone lower bound solution satisfying Pareto-indifference, equal welfares for equal preferences, and consistency.*

Roemer (1986a,b, 1988) formulates consistency requirements with respect to changes in the number of goods. We will not review these papers here as they importantly depend on utility information, which we have chosen to ignore in defining the scope of this survey. (Iturbe-Ormaetxe and Nieto, 1992, 1996b, provide further results along the same lines.)

8.2 Minimal consistent enlargements

When a solution is not *consistent*, one way to evaluate how far from being *consistent* it is. We propose two procedures for doing this.

First, it follows directly from the definitions that the intersection of an arbitrary family of *consistent* solutions, if it constitutes a well-defined solution (that is, if it is non-empty for each economy in its domain), is *consistent*. Also, for most natural ways of specifying allocation problems, the solution that associates with each economy its entire feasible set is *consistent*. Now, given a solution φ, consider the correspondence that associates with each economy its set of allocations that are selected by all of the *consistent* solutions containing φ. Since this family is non-empty, this correspondence is a well-defined solution, and it is clearly the minimal *consistent* solution containing φ, its **minimal consistent enlargement**.

This definition is proposed and explored by Thomson (1994d) who establishes certain algebraic properties of the concept and applies it to examples. The minimal *consistent* enlargement of the union of two solutions is the union of their minimal *consistent* enlargements. The minimal *consistent* enlargement of their intersection is a subsolution of the intersection of their minimal *consistent* enlargements; the inclusion may be strict.

The enlargement is sometimes considerable. For instance, the minimal *consistent* enlargement of the equal-division lower bound and Pareto solution—recall that this solution is not *consistent*—is "essentially" the Pareto solution. Also, that of the Ω-egalitarian-equivalence and Pareto solution is "essentially" the egalitarian-equivalence and Pareto solution.

A second procedure to evaluate how far a solution is from being *consistent* is, provided that it contains at least one *consistent* solution, to reduce it instead of enlarging it. This is because *consistency* is preserved under arbitrary unions. Then, define the **maximal consistent subsolution** of the solution as the union of all of its *consistent* subsolutions. Parallel algebraic relations can be established for the maximal *consistent* subsolution of the intersection and the union of two solutions as a function of the maximal *consistent* subsolutions of the two of them.

This concept allows us to relate different notions that have been discussed separately in the literature. To describe an application, let us first observe that *replication-invariance* is preserved under union too, and so by the same logic, one can define the **maximal consistent and replication-invariant subsolution** of a given solution, provided the solution has at least one subsolution with these properties. Now, the maximal *consistent* and *replication-invariant* subsolution of the equal-division lower bound solution is the solution defined by requiring that each agent should find his bundle at least as desirable as any bundle in the convex hull of the bundles received by all agents (one of the formal definitions of Subsection 4.4). Also, the strict no-envy solution (Subsection 4.4) is nothing other than the maximal *consistent* and *replication-invariant* subsolution of the average no-envy solution (Subsection 4.4).

8.3 Consistency in economies with individual endowments

In economies with individual endowments, formulating *consistency* is not straightforward. If we imagine the departure of some agents with their bundles and keep the endowments of the agents who stay as initially specified, the list of bundles assigned to these agents initially will not be feasible given their endowments.

One possible resolution of this feasibility problem is to adjust the endowments of the agents who stay. Dividing equally among them the difference between the sum of the consumptions and the sum of the endowments of the agents who leave comes to mind (Dagan, 1995; Thomson, 1992), but revised endowments may have negative coordinates, which will require further adjustments.

Another possible resolution is to add to the description of an economy a "gap vector" $T \in \mathbb{R}^{\ell}$: a positive coordinate of T is understood as a surplus of the corresponding good and a negative coordinate as a deficit. A **generalized economy** is a list $(R, \omega, T) \in \mathcal{R}^N \times \mathbb{R}_+^{\ell N} \times \mathbb{R}^{\ell}$ such that $\sum \omega_i + T \geqq 0$, and a feasible allocation for it is a list $(z_i)_{i \in N} \in \mathbb{R}_+^{\ell N}$ such that $\sum z_i = \sum \omega_i + T$. This formulation is proposed by Thomson (1992) and Dagan (1994). Let φ be a solution defined on a domain of generalized economies. To reduce $(R, \omega, T) \in \mathcal{R}^N \times \mathbb{R}_+^{\ell N} \times \mathbb{R}^\ell$ with respect to $N' \subset N$ and $z \in \varphi(R, \omega, T)$, we restrict the preference and endowment profiles to the members of N' and adjust the gap vector by the difference between the sum of the consumptions of the departing agents and the sum of their endowments: this yields the list $(R_{N'}, \omega_{N'}, T + \sum_{N \setminus N'}(\omega_i - z_i))$. Equipped with this notion of a reduction, *consistency* takes the usual form: $z_{N'}$ should be chosen by φ for the reduced economy.

Natural examples of *consistent* Walrasian-like solutions can be based on two alternative choices for the distribution of the gap among agents, equal division on the one hand and proportional division on the other (Thomson, 1992). Peleg (1996) establishes the existence of an extension of the notion of a Walrasian equilibrium for generalized economies. His notion covers the examples just mentioned. He suggests, for each price vector, to add to each agent's income a share of the value of the gap calculated at these prices, in such a way that the agent's total income be a non-negative and continuous function.

It is fair to say that none of the characterizations obtained for these models are as natural as the ones we presented for the model without individual endowments. Indeed, most of them allow solutions to be empty-valued, and in fact, empty-valuedness is frequent (Dagan, 1994; van den Nouweland, Peleg, and Tijs, 1996). If not, they involve strong additional requirements (Korthues, 1996, 2000).

Serrano and Volij (1998) also explore the issue in the context of production economies, and propose two definitions inspired by concepts of cooperative game theory. Their main results are characterizations of the Pareto solution, the core, and the Walrasian solution.

9 Public goods

The formulation and the study of equity criteria in economies with public goods has been the object of much less attention than economies with only private goods.

9.1 Basic solutions

Before beginning our discussion of fairness for this model, we refer the reader to the classical marginal conditions for efficiency identified by Samuelson (1955), and subsequently refined by Saijo (1990), Campbell and Truchon (1988), and Conley and Diamantaras (1996). The topological properties (closedness and connectedness) of the set of efficient allocations are studied by Diamantaras and Wilkie (1996). This set coincides with the set of weakly Pareto efficient allocations under significantly stronger assumptions than in private good economies.

We begin our discussion of fairness by considering the case of one private good and one public good, the public good being produced from the private good according to a linear technology, which, with an appropriate choice of units of measurement, we can assume to be one-to-one. If $|N| = 2$, the feasible allocations can be identified with the points of an equilateral triangle of appropriate size, as in Figure 8 (Kolm, 1970. A pedagogical presentation is Thomson, 1999b). An allocation is a vector $z \equiv (x_1, x_2, y) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ with $x_1 + x_2 + y = \Omega_x$, where the amount $\Omega_x \in \mathbb{R}_+$ of the private good initially available can be consumed as such—this accounts for $x_1 + x_2$ units of it, or used as an input in producing y units of the public good, which requires y units of it. Let $\mathcal{E}^N_{pub}$ denote a class of economies.

Since agents necessarily consume the same amount of the public good, an allocation $z \equiv (x_1, x_2, y)$ is **envy-free** if and only if $x_1 = x_2$. (More generally, as long as there is only one private good, the no-envy and no-domination criteria coincide.) An envy-free allocation can then be equivalently described as an "equal contribution allocation". Thus, the set σ of envy-free allocations is the vertical segment containing the top vertex. It is independent of preferences. (Independence holds whenever there is only one private good.) It intersects the Pareto-optimal set, which typically is a curvi-linear segment with end-points on the slanted axes of the triangle, at a finite number of points. In Figure 8a, the Pareto set is "thick", and there is a continuum of envy-free and efficient allocations.

There are many situations in economic theory where the **Lindahl solution** has been found to be the natural counterpart for public good economies of the Walrasian solution. Given the usefulness of the latter, when operated from equal division, in producing envy-free allocations, one may wonder whether the Lindahl mechanism will be equally useful in achieving this goal. However, since the public good can be produced at several alternative levels,

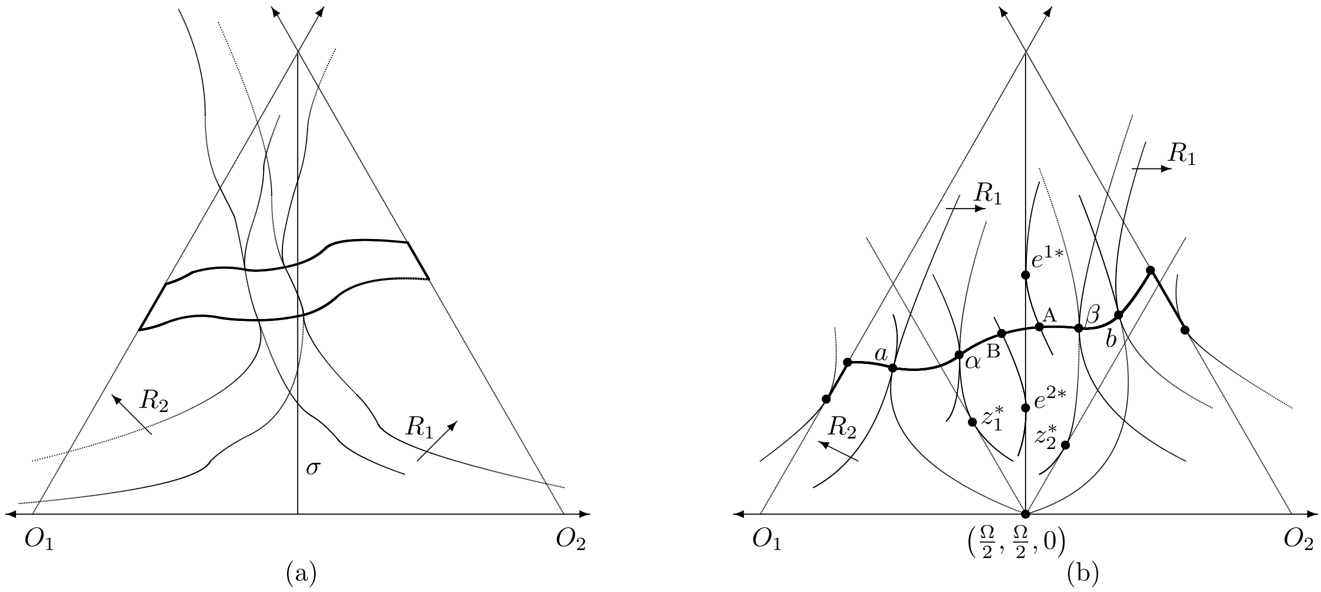


Figure 8: Envy-free allocations in the Kolm triangle. (a) The set of envy-free allocations is the segment σ. The Pareto set is the horizontal wavy band connecting the two slanted sides. (b) The curvilinear segment from α to β is the set of efficient allocations that each agent finds at least as desirable as the best bundle he could achieve if endowed with an equal share of the social endowment of the private good and given unhampered access to the technology. The curvilinear segment from A to B is the set of efficient allocations that each agent finds at most as desirable as the equal-bundle allocation he prefers (e^{1*} for agent 1 and and e^{2*} for agent 2).

there does not exist a unique "point of equal division". In the Kolm triangle, any point of σ qualifies. Given $\omega \in \sigma$, operating the Lindahl solution from ω yields an envy-free allocation only accidentally. Also, the Lindahl solution operated from any $\omega \in \sigma$ does not necessarily treat identical agents identically, (in contrast with the Walrasian solution); at a Lindahl allocation, two identical agents may receive bundles that are not indifferent to each other according to their common preferences. It is true, however, that there always are Lindahl allocations at which identical agents receive equivalent bundles (Champsaur, 1976, studies the continuity properties of the subsolution of the Lindahl solution that selects these allocations).

The above observations indicate that the existence of envy-free and efficient allocations cannot be obtained as a direct corollary of theorems stating the existence of Lindahl allocations (following the pattern we had observed in exchange economics). However, under standard assumptions on preferences and production sets that we will not state in detail, and if for each efficient allocation, the set of allocations that are Pareto-indifferent to it is contractible, and if there are no agent-specific input (these are the critical assumptions), envy-free and efficient allocations exist (Diamantaras, 1992). Moreover, if envy-free and efficient allocations cannot be reached by operating the Lindahl solution from equal division, they are supported as "public competitive equilibria" when taxation is proportional to incomes and endowments are equal (Foley, 1967), because equal expenditure on the private goods obtains then. That every envy-free and efficient allocation is an equilibrium allocation of this type is proved for an economy with a large set of agents modelled as a continuum by Diamantaras (1991).

The notion of **egalitarian-equivalence** is well defined for public good economies, and egalitarian-equivalent and efficient allocations exist under general conditions. Sato (1985) considers the case when there is only one private good and advocates the selection from the egalitarian-equivalence and Pareto solution obtained by requiring the reference bundle to be the unit vector in the private good direction. This choice provides a natural interpretation of the reference bundle as measuring each "agent's willingness to pay for the public good in terms of the private good".

The case of one public good is of particular interest: Suppose there are ℓ private goods, which can be either consumed directly or used in the production of the public good. Let $Y \subseteq \mathbb{R}_+^{\ell+1}$ be the production set. There is a social endowment $\Omega \in \mathbb{R}_+^{\ell}$ of the private goods, and a set N of agents with preferences defined over $\mathbb{R}_+^{\ell+1}$. The following is the selection from the egalitarian-equivalence and Pareto solution obtained by requiring the reference bundle to be of the form $(\frac{\Omega}{|N|}, y_0)$ for some $y_0 \geq 0$.

Definition (Mas-Colell, 1980b) Given $e \equiv (R, \Omega, Y) \in \mathcal{E}_{pub}^N$, let $E_y(e) \equiv \{z \in Z(e)$: there is $y_0 \geq 0$ such that for each $i \in N$, $z_i \; I_i \; (\frac{\Omega}{|N|}, y_0)\}$.

Sato (1987) proposes a notion of equity for which the Lindahl solution plays a role similar to the role played by the Walrasian solution. He also discusses generalizations (1990).

The issue of informational efficiency is addressed by Aizpurua and Manresa (1995) for a general model with arbitrarily many private and public goods, but in which there are no agent-specific inputs. A variant of the Lindahl mechanism that they introduce under the name of "Lindahl egalitarian" (which produces the public competitive equilibria described earlier) has minimal dimensionality among all mechanisms satisfying a regularity condition and realizing envy-free and efficient allocations.

Next, we formulate upper and lower bounds similar to the ones defined earlier for economies with private goods (Section 5.2).

The first bound, the **equal-division lower bound**, is simply that each agent should be made at least as well off as he would be by consuming an equal share of the social endowment of the private good(s). In the example depicted in Figure 8, the unique envy-free and efficient allocation meets this bound, but it is easy to modify it to show that there may be no envy-free and efficient allocation that does.

A second lower bound on an agent's welfare is obtained by imagining that he is alone and has to cover the full cost of the public goods.[42]

Definition (Moulin, 1992c) Given $e \equiv (R, \Omega, Y) \in \mathcal{E}^N_{pub}$, the allocation $z \in Z(e)$ meets the **equal-division free-access lower bound for *e*** if for each $i \in N$, $z_i \ R_i \ z_i^*$, where z_i^* is any bundle that would maximize agent i's welfare if endowed with an equal share of the social endowment and given free access to the technology.

A third bound—this time it is an upper bound—is obtained by first imagining, for each agent, that all others have preferences identical to his, and in this economy of identical agents, imposing efficiency and *equal treatment of equals*. Returning to the actual economy, we require that each agent should be made at most as well off as he would be in this hypothetical economy:

Definition (Moulin, 1992c) Given $e \equiv (R, \Omega, Y) \in \mathcal{E}^N_{pub}$, the allocation $z \in Z(e)$ meets the **identical-preferences upper bound for *e*** if for each $i \in N$, $z_i^* \ R_i \ z_i$, where z_i^* is any bundle that agent i would be assigned by any efficient solution satisfying *equal treatment of equals* in the hypothetical economy in which each other agent had preferences identical to his.

The existence of efficient allocations meeting the *identical-preferences upper bound* is guaranteed under general assumptions (Moulin, 1992c).

On the domain of one-input one-output economies, the *identical-preferences upper bound* and the *equal-division free-access lower bound* are compatible. Here is an example of an efficient solution satisfying both:

Definition (Moulin, 1992c) Let $e \equiv (R, \Omega, Y) \in \mathcal{E}^N_{pub}$ be a one-input one-output economy. The allocation $z \in Z(e)$ is an **equal-ratio–equivalent allocation for *e*** if there is $\lambda \in \mathbb{R}_+$ such that for each $i \in N$, $z_i \ I_i \ z_i^*$, where

[42]The previous bound is often called "individual rationality from equal division" and the next one, "strong individual rationality from equal division".

z_i^* is any bundle that would maximize agent i's welfare if given access to an equal share of the social endowment and to the technology $Y^\lambda \equiv \{(x, y) \in \mathbb{R}^2 \colon (\lambda x, y) \in Y\}$.

The free-access constraints can be generalized to groups:

Definition (Moulin, 1992c) Given $e \equiv (R, \Omega, Y) \in \mathcal{E}^N_{pub}$, the allocation $z \in Z(e)$ is in the **equal-division free-access core of** $\boldsymbol{e}$ if for each $S \subseteq N$, there is no list $(z_i^*)_{i\in S}$ that is feasible for the coalition S (without the contribution of the complementary coalition), if endowed with $\frac{|G|}{|N|}\Omega$ and given unhampered access to the technology, and such that for each $i \in S$, $z_i \; R_i \; z_i^*$, and for at least one $i \in S$, $z_i \; P_i \; z_i^*$.

Under general conditions, this solution is non-empty: Indeed, the solution $E_y P$ is a subsolution of the equal-division free-access core. The solution $E_y P$ also happens to satisfy very appealing monotonicity properties in response to improvements in the technology (Section 9.4).

In a class of economies with one private good and one public good produced according to a convex technology, further logical relations between the identical-preferences upper bound and the equal-division free-access lower bound are developed by Watts (1999). She shows that, except in trivial cases, the former does not imply the latter.

9.2 Notions of equal, or equivalent, opportunities

The notions of equal opportunities introduced in Section 6 can be adopted to the current situation, but not all the results established in the private good case extend. In particular, even if $|N| = 2$, there is no family of choice sets whose associated equal-opportunity solution is a subsolution of the Pareto solution. Suppose now that preferences satisfy the classical assumption and that the technology is linear. Then, if $|N| = 2$, the equal-opportunity–equivalence solution associated with the family of linear choice sets is the egalitarian-equivalence solution. If $|N| > 2$, there is no necessary containment between the intersections of these two solutions with the Pareto solution, a result that mirrors one obtained for the private good case. Most importantly, under appropriate assumptions on preferences, the intersection of the equal-opportunity–equivalent and Pareto solution associated with the family of choice sets normal to a fixed price vector is well-defined.

9.3 Social endowment monotonicity

Monotonicity questions are addressed in the context of public good economies by Thomson (1987c) and Moulin (1992c). The interesting case here is when there is, in addition to the public goods, only one private good. Indeed, as soon as there are two private goods, the impossibility results obtained for exchange economies extend to this more general class of economies. It suffices to consider "degenerate" public good economies in which agents happen to only care about the private goods.

If there is only one private good, and preferences are such that the r-egalitarian-equivalence and Pareto solution E_rP, where r is the unit vector corresponding to that good, is well defined (this requires that all indifference surfaces intersect the axis corresponding to the good), then this solution is *resource-monotonic*. Therefore, *resource-monotonic* selections from the egalitarian-equivalence and Pareto solution exist. However, no such selections from the no-envy and Pareto solution exists:

Theorem 9.1 (Thomson, 1987c) Domain: one private good and one public good; strictly monotonic and convex preferences; linear technologies. *No selection from the no-envy and Pareto solution is resource monotonic.*

9.4 Technology-monotonicity

Here, we address the issue of *technology-monotonicity.* Recall the definition of the selection from the egalitarian-equivalence and Pareto solution obtained by requiring the reference bundle to be of the form $(\frac{\Omega}{|N|}, y_0)$. Given $e \equiv (R, \Omega, Y) \in \mathcal{R}^N \times \mathbb{R}_+ \times \mathcal{Y}$, $E_y(e) \equiv \{z \in Z(e)$: there is y_0 such that for each $i \in N$, $z_i \ I_i \ (\frac{\Omega}{|N|}, y_0)\}$. We have the following characterization of this solution, in which assumptions are made to guarantee that it is well-defined:

Theorem 9.2 (Moulin, 1987a) Domain: one private good and one public good; the preferences of each agent i, defined over a set of the form $[0, \omega_i] \times \mathbb{R}_+$, are strictly monotonic with respect to the private good, weakly monotonic with respect to the public good, and such that for each $x \geq 0$, there is a unique $z_i \in Y_i$ for which $(x, y_i) \ I_i \ (\omega_i, 0)$; technology exhibits returns-to-scale that are bounded above. *The solution E_yP is the only selection from the equal-division free-access lower bound and Pareto solution satisfying technology-monotonicity.*

The assumption on the technology can be relaxed. What is important is that production sets be closed under union. However, the assumption that there is only one public good cannot be removed (Ginés and Marhuenda, 1996, 1998).

The solution characterized in Theorem 9.2 is applied by Weber and Wiesmeth (1991). These authors identify assumptions on preferences under which it actually coincides with the Lindahl solution (these assumptions are obviously quite strong), and they define a generalization of it in Weber and Wiesmeth (1990).

9.5 Welfare-domination under preference-replacement

Here, we will only note that the selection from the egalitarian-equivalence and Pareto solution characterized in Theorem 9.2 happens to be such that any change in the preferences of one agent affects all other agents in the same direction. Moreover, although we only required that an improvement in the technology should make all agents at least as well off as they are initially, it turns out that any change in the technology affects the welfares of all agents in the same direction (that is, even when the old and the new technologies cannot be ranked, the welfare levels can).

We omit the formal statement of *welfare-domination under preference-replacement*, and simply note that a counterpart of Theorem 7.2 holds for this domain.

9.6 Monotonicity in economies with individual endowments

Here, we consider economies in which agents may be individually endowed, and discuss properties introduced in Subsection 7.5. We ask whether it is possible to ensure that an increase in an agent's endowment never hurts any of the others. We have the following:

Theorem 9.3 (Thomson, 1987b) Domain: economies one private good and one public good; monotonic and convex preferences; convex technologies. *No selection from (a) the no-envy and Pareto solution or (b) the egalitarian-equivalence and Pareto solution, satisfies* no negative effect on others.

Under certain properties of preferences, the Lindahl solution is *population-monotonic* (Sertel and Yıldız, 1998).

9.7 Population-monotonicity

Turning now to variable populations with fixed resources, we have a possibility result provided we are satisfied with weak Pareto-optimality. Let $e \equiv (R, \Omega, Y) \in \mathcal{E}^N_{pub}$ and $\underline{B}_{ed}(e)$ be the set of feasible allocations for e at which each agent is at least as well off as he would be at the best point he could achieve, if endowed with an equal share of the social endowment of the private good and given unhampered access the technology: this is the equal-division free-access lower bound solution.

In situations in which sufficiently strong positive external effects exist, the natural monotonicity requirement to formulate in response to increases in the population is that all agents initially present should be affected positively. Let us refer to this condition as **population-monotonicity$_+$**.

The selection from the egalitarian-equivalence and Pareto solution obtained by requiring the reference bundle to be the unit vector corresponding to the public good is a *population-monotonic$_+$* selection from the Pareto solution.

In the quasi-linear case, the Shapley value (Shapley, 1953) when applied to the free-access game associated with each economy induces a *population-monotonic* solution. A *population-monotonic* solution meeting the *identical-preferences upper bound* is constructed by Moulin (1990a). Parallel results hold for the case of bads (Moulin, 1992a).

Let us say that a solution is **weakly population-monotonic** if when new agents come in, all agents initially present are affected in the same direction.

Proposition 9.1 (Thomson, 1987c) Domain: one private good and possibly multiple public goods; strictly monotonic preferences except that any bundle that is not strictly positive is indifferent to 0. *There are selections from the equal-division free-access lower bound solution satisfying* weak population-monotonicity.

A counterpart of Theorem 7.9 holds for economies with one private good and possibly multiple public goods, the population of agents being modelled as an atomless continuum. It is a characterization of the solution that selects any efficient allocation such that there is a ratio $a \geq 0$ with the property that each agent is indifferent between his bundle and the best bundle he could reach among all bundles obtained for which he would have to pay the fraction a of the cost of production. Let us call this solution the **equal-factor-equivalence solution**.

Theorem 9.4 (Sprumont, 1998) Domain: one private good and possibly multiple public goods; continuum of agents; strictly monotonic and convex preferences; cost function for the public goods is strictly increasing, strictly convex, takes value 0 at 0, and satisfies a mild regularity condition. *The equal-factor–equivalence and Pareto solution is the only selection from the identical-preferences upper bound and Pareto solution satisfying welfare-domination under preference-and-population–replacement.*

As for the private good version of the theorem (Theorem 7.9), uniqueness depends on the possibility of varying preferences and populations jointly. A version of the result holds for finite economies provided a certain form of *replication-invariance* is imposed too.

9.8 Consistency

In a public good economy, conceptual problems arise in expressing the idea of consistency (Section 8) because one cannot imagine the departing agents to leave with their consumptions. Indeed, the public good components of their consumptions are also consumed by the agents who stay. Proposals have been made to deal with this problem but the definitions are not as compelling as in the private good case.

For a version of the model in which the set of agents is represented as a continuum, Diamantaras (1992) provides a characterization of the public competitive equilibrium solution on the basis of such a notion.

Van den Nouweland, Tijs, and Wooders (2002) characterize the generalization of Lindahl equilibrium proposed by Kaneko (1977a,b) under the name of "ratio equilibrium". In defining a reduced economy, they make adjustments in the cost function as a function of the ratios associated with the allocation that is the point of departure.

10 Indivisible goods

Estate or divorce settlements often involve items that cannot be divided (houses, family heirlooms), or can only be divided at a cost that would make the division undesirable (silverware). Other examples that have been the subject of much discussion recently are positions in schools or organs for transplants patients. In this section, we reconsider the theory of fair allocation in the presence of such indivisible goods, called "objects". We assume that there is also an infinitely divisible good, called "money". We focus on situations in which each agent can consume at most one object. An illustration is the problem of allocating rooms to students in a house they share, and specifying how much each of them should contribute to the rent. The multi-object-per-agent case and the object-only case are discussed in Subsection 10.7.

Some, but not all, of the concepts introduced earlier can be adapted to this situation, but interestingly, due to the special structure of the model, several equivalences exist that do not hold in general. Consistency properties can be met and characterizations are available based on them. On the other hand, monotonicity properties are quite restrictive and hold only on narrow subdomains.

10.1 The model

Let Ω be a social endowment consisting of some amount M of money and a set A of objects. This endowment is to be distributed among a set N of agents. Unless specified otherwise, $|N| = |A|$. Each agent $i \in N$ has preferences R_i defined over $\mathbb{R} \times A$ (or over $\mathbb{R}_+ \times A$, but for these introductory paragraphs, we choose the former). They are continuous and strictly monotonic with respect to money, and satisfy the following "compensation assumption": for each bundle $(m_i, \alpha) \in \mathbb{R}\times A$, and each object $\beta \in A$, there is $m'_i \in \mathbb{R}$ such that $(m'_i, \beta)\ I_i\ (m_i, \alpha)$. Let $\mathcal{R}_{ind}$ be the class of all such preferences. An **economy** is a list $e \equiv (R, M, A)$ as just described, and $\mathcal{E}^N_{ind}$ is our generic notation for a class of economies. A **feasible allocation** for $e \equiv (R, M, A) \in \mathcal{E}^N_{ind}$ is a pair $z \equiv (m, \sigma)$ consisting of a vector $m \in \mathbb{R}^N$ such that $\sum m_i = M$ specifying how much money each agent receives and a bijection $\sigma\colon N \to A$ specifying which object is assigned to each agent. The bundle received by agent $i \in N$ at z is $z_i \equiv (m_i, \sigma(i))$. Let $\boldsymbol{Z(e)}$ denote the set of feasible allocations of $e \in \mathcal{E}^N_{ind}$.

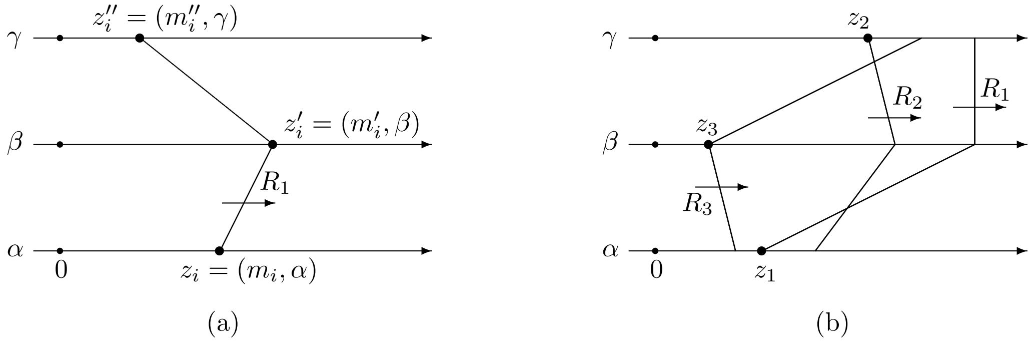


Figure 9: The no-envy solution applied to economies with indivisible goods. (a) Representation of consumption space and of an indifference curve for agent 1. (b) The allocation z is not envy-free since agent 3's indifference curve through z_3 passes to the left of z_1.

Situations where there are fewer objects than agents are accommodated by introducing a "null object", denoted ν, the objects in A being then referred to as "real objects" when there is a risk of ambiguity. There are always enough copies of the null object for each agent to end up with one object. Denoting by A^* the augmented set $A \cup \nu$, preferences are then defined over $\mathbb{R} \times A^*$. A real object α is **desirable** for R_i if for each $m_i \in \mathbb{R}$, (m_i, α) R_i (m_i, ν). It is **undesirable** if preference always goes in the other direction. An object may of course be neither desirable nor undesirable. If there are fewer agents than objects, some objects are unassigned. In some applications, it is natural to require that the null object not be assigned until all real objects are, even if these objects are undesirable. For instance, they could be tasks to be assigned to students in the house they share. Some students may find some of these tasks desirable and others not (cooking), but all tasks may have to be carried out.

Figure 9a gives a convenient graphical representation of the model for $|N| = |A|$. There are $|A|$ axes indexed by the elements of A. Along the axis indexed by each object is measured the amount of the divisible good that comes with that object. The broken line through $z_i \equiv (m_i, \alpha)$ links z_i to two other bundles $z_i' \equiv (m_i', \beta)$ and $z_i'' \equiv (m_i'', \gamma)$ that agent i finds indifferent to z_i; it can be thought of as an indifference curve. **Quasi-linearity** of preferences means that indifference curves are all obtained from any one of them by horizontal translations.

A variant of the model just described is when the objects in A are identical. For instance, they may be jobs on an assembly line. When there are more workers than jobs, the null object is interpreted as being unemployed. Finally is the even more special situation in which $|A| = 1$. In either one of these cases, any recipient of a real object is called a "winner". The others are called "losers".

10.2 Basic solutions

The **no-envy** concept applies directly to this model. In Figure 9b, agent 1 envies no one, agent 2 envies no one, but agent 3 envies agent 1.

For the distribution of multiple copies of the same object, in order for the winners not to envy each other, they should receive equal amounts of money. For the losers not to envy each other, they should also receive equal amounts of money. In addition, each winner should find the "winning bundle" at least as desirable as the "losing bundle", and each loser should find the "losing bundle" at least as desirable as the "winning bundle".

It is clear that if consumptions of money have to be non-negative, envy-free allocations may not exist. Imagine for instance that all agents have the same preferences and the social endowment of money is 0. Conversely, one may hope that if the social endowment of money is sufficiently large, it is possible to compensate those agents who do not receive the objects they would prefer. This hope is justified:

Theorem 10.1 (Svensson, 1983) Domain: one infinitely divisible good, and a set A of objects; equal number of agents; preferences, defined on $\mathbb{R}_+ \times A$, are strictly monotonic with respect to money, and such that for each $i \in N$ and each $\alpha \in A$, there is an allocation at which the bundle containing α is most preferred among all bundles the allocation is composed of. (a) *Envy-free allocations exist.* (b) *Envy-free allocations are efficient.*

Techniques rather different from Svensson's have been used to prove the existence of envy-free allocations (Maskin, 1987; Alkan, Demange, and Gale, 1991). When consumptions of money are unbounded below, and if preferences are continuous, monotonic with respect to money, and satisfy the compensation assumption, envy-free allocations exist without any assumption relating preferences and the social endowment of money (Alkan, Demange, and Gale, 1991). A constructive proof is possible that covers preferences that need not be monotonic but otherwise satisfy assumptions similar to those of Theorem 10.1 (Su, 1999). For a general proof that covers all of the above, see Velez (2007). When the objects are identical, an elementary existence proof is also available (Tadenuma and Thomson, 1993).

The implication stated as Part (b) of Theorem 10.1 fails if $|A| > |N|$. In that case, no-envy may well be achieved by assigning objects that all agents find inferior to objects that are not assigned. However, the implication can be recovered under a certain strengthening of the definition of no-envy, a definition that is non-vacuous under general conditions (Alkan, Demange and Gale, 1991). The distributional merits of the stronger notion can be questioned however, as it sometimes seems to favor particular agents (Tadenuma, 1994).

A variant of the model is when each agent can consume only some of the objects available (Svensson, 1988). To illustrate, the objects could be jobs, and not all agents may be qualified for all jobs. Svensson states assumptions guaranteeing the existence of envy-free allocations in this context.

For quasi-linear preferences, several algorithms leading to envy-free allocations have been developed. Then, and except for degenerate cases, the assignments of objects are the same at all efficient allocations (efficiency is undisturbed by transfers of money among agents), so that one can speak of an "efficient allocation of objects". There are finitely many assignments of objects, so the efficient ones can be identified by exhaustive search. Aragones (1995)'s starting point is an efficient assignment. She considers the case when consumptions of money are non-negative and her algorithm identifies the smallest social endowment of money M^* guaranteeing the existence of envy-free allocations. This amount depends of course on preferences. The envy-free allocation obtained then is unique up to Pareto-indifference, and it provides the basis for the definition of a selection from the set of envy-free allocations of that economy when the social endowment of money M is at least M^*, by dividing equally among all agents the difference $M - M^*$.

Klijn's algorithm (2000) starts from an arbitrary feasible allocation. Envy cycles are first eliminated. If an envy relation remains, transfers of money from the envied agent to the envious agent are made to eliminate it but additional transfers to or from other agents are carried out too in order to ensure that no new envy relation is created.[43]

At each step of the market-like algorithm proposed by Abdulkadiroğlu, Sönmez, and Ünver (2004), no-envy is met, as all agents maximize their preferences over a common budget set, but feasibility is not, and convergence is to a feasible allocation. When envy-free allocations exist at which consumptions of money are all non-positive, as might be needed when the social endowment of money is negative, objects are desirable, and each agent is required to pay something for receiving an object (think of the rent division application), the algorithm produces such an allocation. A family of

[43]The algorithm can be modified to produce the extreme points of the set of envy-free allocations.

algorithms in that spirit is developed by Ünver (2003).[44]

The notion of **egalitarian-equivalence** also applies directly to this model. Egalitarian-equivalent and efficient allocations exist very generally, when consumptions of money are unbounded below and the compensation assumption holds. The proof is similar to that of the existence of r-egalitarian-equivalent and efficient allocations in classical economies. When preferences are defined over $\mathbb{R}_+ \times A$, existence holds under similar assumptions as the ones guaranteeing that of envy-free allocations in Theorem 10.1. In either case, to each object can be associated a reference bundle containing that object to which corresponds an egalitarian-equivalent and efficient allocation (Svensson, 1983b). Thus, there are as many egalitarian-equivalent and efficient allocations as objects. Let $e \equiv (R, M, A) \in \mathcal{E}^N$. Figure 10a illustrates that if $|N| = 2$ and $z \in EP(e)$, then $z \in F(e)$. If $z \in EP(e)$ with reference bundle z_0, then there is $i \in N$ such that $z_i = z_0$, so that agent i envies no one, but if $|N| \geq 3$, there may be occurrences of envy. Moreover, just as in the classical case, there are economies in which all egalitarian-equivalent and efficient allocations violate no-envy (Thomson, 1990a).

If there is only one real object, these notions are compatible however. Consider the solution $\boldsymbol{F^*}$ that selects the envy-free allocation that is the least favorable to the winner: at this allocation, the winner is indifferent between his bundle and the losers' common bundle. (For some configurations of preferences, there are several, Pareto-indifferent, allocations with this property.) This allocation is egalitarian-equivalent, with the losers' bundle serving as reference bundle.

The **Walrasian solution** can be adapted to the present model as follows: for each $\alpha \in A$, let $p_\alpha \in \mathbb{R}_+$. We call p_α the **price** of object α. A **price vector** is a list $p \equiv (p_\alpha)_{\alpha \in A}$.

Definition Given $e \equiv (R, M, A) \in \mathcal{E}^N_{ind}$, the allocation $z \in Z(e)$ is an **equal-income Walrasian allocation for *e***, written as $\boldsymbol{z \in W_{ei}(e)}$, if there are $p \in \mathbb{R}^A_+$ and $M_0 \in \mathbb{R}_+$ such that for each $i \in N$ and each $z'_i \equiv (m'_0, \alpha') \in \mathbb{R} \times A$ satisfying $m'_0 + p_{\alpha'} \leq M_0$, we have $z_i \; R_i \; z'_i$.

The definition is illustrated in Figure 10b. Recall that in classical economies, the equal-income Walrasian allocations constitute a small subset

[44] Another procedure is defined by Brams and Kilgour (2001) that generates an efficient allocation at which agents do not receive money, but it may not be envy-free.

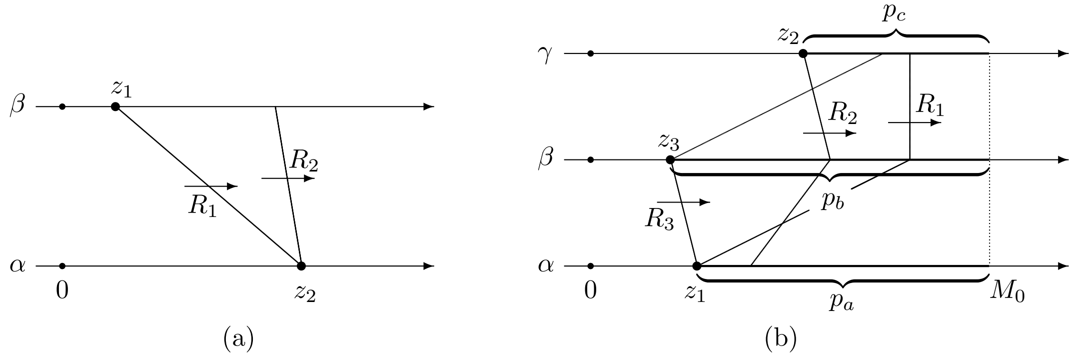


Figure 10: Egalitarian-equivalence and the equal-income Walrasian solution in economies with indivisible goods. (a) For $|N| = 2$, if z is egalitarian-equivalent and efficient, it is envy-free. (b) Independendly of the number of objects, if z is an equal-income Walrasian allocation, it is envy-free.

of the set of envy-free and efficient allocations. Here, the two notions coincide. We also have equivalence between no-envy for individuals and **group no-envy** (defined as in the classical case):

Theorem 10.2 (Svensson, 1983b) Domain: one infinitely divisible good, and a set A of objects; preferences, defined on $\mathbb{R}_+ \times A$ (or on $\mathbb{R} \times A$), are monotonic with respect to the divisible good. (a) *An allocation is envy-free if and only if it is an equal-income Walrasian allocation.* (b) *An allocation is envy-free if and only if it is group envy-free.*

A variety of selections from the no-envy solution have been proposed. First, define the worse-off agent at an allocation as the agent who receives the smallest amount of money. The **maximin money** rule selects the envy-free allocation at which this amount is as large as possible. If the null object is present, define the worse-off agent as the one whose money-only equivalent bundle contains the smallest amount of money. The **maximin money-only-equivalent** rule selects the envy-free allocation at which this amount is as large as possible. Two solutions can be defined in a symmetric way based on identifying the agent whose consumption of money is the largest. The **minimax money** rule selects the envy-free allocation at which the agent who receives the largest amount of money receives the smallest such amount. The **minimax money-only-equivalent** rule performs the parallel exercise with the money-only equivalent bundles (Alkan, Demange, and Gale, 1991. The first criterion is also studied in the quasi-linear case by Aragones, 1995).

Next, (and whether or not the null object is available,) given any envy-free allocation, for each two-agent subgroup and each agent in the pair, calculate the amount of money that should be added to the other agent's bundle for the first agent to be indifferent between his bundle and the second agent's revised bundle. These $|N|(|N| - 1)$ "compensation" terms give a picture of how well each agent is treated in relation to each other agent at the allocation. Then, select the allocation(s) at which all agents are treated "as equally as possible". This can be done in several ways. One is to calculate for each agent the average of his compensation terms over the pairs to which he belongs, and choosing the allocation(s) at which the smallest (across agents) such average is maximal. This **maximin average compensation** rule is *essentially single-valued* (Tadenuma, 1989). Criteria based on lexicographic operations are also possible, focused on the agents who are treated the worse according to the size of their compensation terms (Tadenuma and Thomson, 1995).

Another idea exploits the fact that if $|N| = 2$, the assignment of objects is the same at all envy-free allocations (except for degenerate cases). Given an envy-free allocation, for each two-agent subgroup, identify the two extreme allocations obtained by transferring money from one agent to the other without violating envy. The difference between the amounts of money received by either one of the two agents at these two allocations can be thought of as an "equity surplus" at the allocation. Calculate the share of this surplus that each agent receives. Then, select the allocation(s) at which agents are treated as equally as possible. Again, several choices are possible here depending upon whether an average of these surplus shares is considered, yielding what could be called a **maximin average share** rule, or whether a lexicographic operation is performed on these surplus terms (these last two proposals are developed by Tadenuma and Thomson, 1995). In the two-agent case, and when preferences are quasi-linear, several of these proposals agree.

In the model under study here, dividing resources equally is not an option but an earlier distributional requirement that remains meaningful is that each agent should be made at least as well off as he would be at the (essentially) unique envy-free allocation of the hypothetical economy in which everyone had his preferences. This is the **identical-preferences lower bound**. If $|N| = 2$, meeting this bound is actually equivalent to no-envy, but if $|N| > 2$, the identical-preferences lower bound is weaker (Bevia, 1996a). (In particular, it does not imply efficiency.) Thus, this concept gives us another chance of obtaining positive results when no-envy is too demanding. Unfortunately, there are quasi-linear economies with equal numbers of objects and agents in which all egalitarian-equivalent and efficient allocations violate not only no-envy, as we already know, but in fact the identical-preferences lower bound.

When there are more objects than agents, an allocation may be envy-free and efficient without meeting the identical-preferences lower bound, but it meets the variant of the lower bound obtained by using only the objects that are assigned. Let us call it the **weak identical-preferences lower bound**. Our earlier result concerning the incompatibility of egalitarian-equivalence and the identical-preferences lower bound persists however, since it can be proved by means of an example with an equal number of objects and agents, for which the two versions of the bound coincide (Thomson, 2003b).

10.3 Resource-monotonicity

Next, we consider changes in resources. We begin with the requirement that as the amount of money available increases, all agents should end up at least as well off as they were initially:

Money-monotonicity: For each $(R, M, A) \in \mathcal{E}^N_{ind}$, each $z \in \varphi(R, M, A)$, each $M' > M$, each $z' \in \varphi(R, M', A)$, we have $z' \; R \; z$.

Any selection from the egalitarian-equivalence and Pareto solution obtained by fixing the reference object is *money-monotonic*. If there is only one real object, and if the reference object is chosen to be the null object, we obtain the solution F^*, introduced earlier, which is also a selection from the no-envy solution. This solution enjoys other desirable properties, as we will see.

In the multiple-object case, selections from the no-envy solution exist that are *money-monotonic* (Alkan, Demange, and Gale, 1991).

Next, we require that when additional objects become available, all agents should end up at least as well off as they were initially. This property makes sense if the objects are desirable or if they do not have to be assigned. Of course, in specifying an economy, we now have to allow the numbers of objects and agents to differ. Then, an envy-free allocation is not necessarily efficient and we explicitly impose efficiency. Recall that in our basic definition of an economy, preferences are defined over the cross-product of $\mathbb{R}$ with the set of objects. In specifying the economy that results after the disappearance of some objects, we therefore restrict preferences to the cross-product of $\mathbb{R}$ with the set of remaining objects.

Object-monotonicity: For each $(R, M, A) \in \mathcal{E}^N_{ind}$, each $z \in \varphi(R, M, A)$, each $A' \subset A$, each $z' \in \varphi(R_{|\mathbb{R}\times A'}, M, A')$, we have $z \; R \; z'$.

A negative result holds, even on the quasi-linear domain:

Theorem 10.3 (Alkan, 1994) Domain: one infinitely divisible good and a set A of objects and its subsets; preferences, defined on $\mathbb{R}\times A$, are quasi-linear. *No selection from the no-envy and Pareto solution is object-monotonic.*

A weaker monotonicity requirement can be formulated, which says that for each economy, there should be at least one allocation such that, upon the addition of one more object, an improvement in the welfares of all agents can be achieved. A limited sense in which this requirement of **local extendability in an object-monotonic way** can be met within the no-envy solution is discussed by Alkan, Demange, and Gale (1991). The most general result along these lines covers variations in populations as well, and we find it convenient to wait until our discussion of this issue to give a single statement (Subsection 10.5).

Consider situations in which all real objects have to be assigned before any null object is, independently of whether they are desirable. For instance, objects may be activities that some agents enjoy and others do not, but these activities have to be carried out if there are enough agents for that. Even if preferences are quasi-linear, no selection from the weak identical-preferences lower bound and Pareto solution is **weakly object monotonic**, that is, such that the welfares of all agents are always affected in the same direction by an enlargement of the set of objects (Thomson, 2003b).

We have one positive result to report, which is parallel to Theorem 7.1 pertaining to the classical model. Consider the quasi-linear domain when all objects are desirable. The Shapley value (Shapley, 1953), when applied to the free-access coalitional game associated with each economy, induces an *object-monotonic* selection from the identical-preferences lower bound and Pareto solution (Moulin, 1992a).

10.4 Welfare-domination under preference-replacement

Here, we turn to the requirement introduced in Subsection 7.2 under the name of *welfare-domination under preference-replacement*, that a change in some agent's preferences should affect all other agents in the same direction.

In the one-object case, the property can be met, but in the presence of no-envy, in a unique way, by the solution F^* of Subsection 10.2 (which selects the envy-free allocation that is the least favorable to the winner). This solution is also a selection from the egalitarian-equivalence solution (recall that in the one-object case egalitarian-equivalence and no-envy are compatible). We state this uniqueness result for general preferences, although it also holds on the quasi-linear domain:

Theorem 10.4 (Thomson, 1998) Domain: one infinitely divisible good and a set A consisting of a single object; at least 3 agents; preferences, defined on $\mathbb{R} \times A$, satisfying the compensation assumption. *The solution F^* is the only selection from the no-envy solution satisfying* Pareto-indifference *and* welfare-domination under preference-replacement.

In the case of more than one object, we have an impossibility, even on the quasi-linear domain:[45]

Theorem 10.5 (Thomson, 1998) Domain: one infinitely divisible good and a set A of objects; quasi-linear preferences, defined on $\mathbb{R} \times A$, satisfying the compensation assumption. *No selection from the no-envy solution satisfies* welfare-domination under preference-replacement.

10.5 Population-monotonicity

Next, we consider variations in populations, generalizing the model and the notation in the following way: an economy is a triple (R, M, A) where $M \in \mathbb{R}$ is an amount of money, A is a list of objects, and $R \equiv (R_i)_{i \in N}$, for some $N \in \mathcal{N}$, is a list of preferences defined on $\mathbb{R} \times A$.

A first requirement in this context is that if the social endowment of money is non-negative and the objects are all desirable, none of the agents initially present should benefit from the arrival of additional agents.

We start with the one-object case. First, even when preferences are quasi-linear, *population-monotonicity* is incompatible with no-envy (Alkan; 1994, Moulin, 1990b). In fact, an agent could be better off at any envy-free allocation than if he were alone, so that a violation of the *free-access upper bound* (see Section 7) is unavoidable if no-envy is insisted upon.

However, in this model in which there is no lower bound on consumption spaces, requiring agents to end up at most as well off when new agents come

[45] This result is proved for any $n \geq 4$. Gordon (2000) shows that it also holds for $n = 3$.

in may not be the right thing to do. This is because the model is then essentially equivalent to a production model. Receiving the object is similar to being given a chance to produce. When new agents come in with "good" production functions, they may be able to use the input very productively, and the agents originally present may be made to benefit from it. To be ready to deal with that case and with the case when the new agent has a poor production function, we return to *weak population-monotonicity* (the requirement that changes in population should affect the welfares of all agents who are present before and after the change in the same direction.)

It is easy to see that the solution F^* enjoys the property, but it is essentially the only selection from the no-envy solution to do so. To formally state this characterization, we need the following very mild condition of **neutrality**: if an allocation obtained by exchanges of bundles from one that is chosen by the solution leaves unaffected the welfares of all agents, then it should also be chosen by the solution. We also impose **translation invariance**: for each $t \in \mathbb{R}$, if each preference map is translated by t and the social endowment of money is changed by t times the number of agents, then the recommended bundle for each agent should be obtained from his old one by changing its money component by t units.

Theorem 10.6 (Tadenuma and Thomson, 1993) Domain: one infinitely divisible good and a set A consisting of a single object; preferences, defined over $\mathbb{R} \times A$, are strictly monotonic with respect to money and satisfy the compensation assumption. *The solution F^* is the only selection from the no-envy solution to be neutral, translation invariant, and weakly population-monotonic.*

The next few results pertain to economies in which the object is desirable: an agent would always need to be compensated to give it up. Then, in economies with quasi-linear preferences in which the amount of money is 0, a *population-monotonic* selection from the identical-preferences lower bound and Pareto solution can be defined that differs from F^* (Moulin, 1990b). For each economy, the allocation it recommends is the one obtained by applying the Shapley value to the associated free-access game. When preferences are not necessarily quasi-linear and the amount of money is non-negative, the existence of a solution enjoying these same properties can still be demonstrated (Bevia, 1996c). Its restriction to the quasi-linear case is the solution induced by the Shapley-value in the manner described above.

In the multiple-object case, the selection from the egalitarian-equivalence and Pareto solution obtained by requiring the reference bundle to contain a fixed object is *weakly population-monotonic*, but it is not guaranteed to be a selection from the no-envy solution anymore. In fact, if no-envy is imposed, we have the following impossibility, which holds even on the quasi-linear domain:

Theorem 10.7 (Tadenuma and Thomson, 1995) Domain: one infinitely divisible good and a set A of objects; preferences, defined over $\mathbb{R}\times A$, are strictly monotonic with respect to money and satisfy the compensation assumption. *No selection from the no-envy solution is weakly population-monotonic.*

The following positive result is available. Consider the quasi-linear domain and suppose that the social endowment of money is 0. The Shapley value, when applied to the free-access coalitional game associated with each economy, induces a *population-monotonic* selection from the identical-preferences lower bound and Pareto solution (Moulin, 1992a).

Theorem 10.7 shows that *weak population-monotonicity* is a very strong requirement in the present context and it is therefore natural to investigate the possibility of satisfying weaker requirements. The following question can be asked (Alkan, 1994): for each economy, and when all objects are desirable, *is there an allocation* such that upon the arrival of an additional agent, all agents initially present can be made at most as well off as they were initially, and such that upon the departure of an agent, all remaining agents can be made at least as well off as they were initially? If yes, the allocation is **locally lower-extendable, and locally upper-extendable (respectively), in a population-monotonic way**. This definition extends a notion introduced in Section 10.3 in connection with variations in the number of objects. Alkan (1994) considers a two-part definition that covers both variations in objects and variations in populations. We refer to it by the shorter phrase **locally upper, or lower, extendability**. It turns out that if no-envy is required, only a limited form of these properties holds:

Theorem 10.8 (Alkan, 1994) Domain: one infinitely divisible good and a set A of objects and subsets; preferences, defined over $\mathbb{R} \times A$, are strictly monotonic with respect to money and satisfy the compensation assumption. (a) *When objects are desirable, the minimax money allocation of each economy is an envy-free and locally upper-extendable allocation.* (b) *The* *maximin money-only-equivalent allocation of each economy is locally lower-extendable.* (c) *When objects are desirable and there are at least as many agents as objects, the minimax money allocation is the only locally upper-extendable in an object-monotonic way envy-free allocation.*[46]

Local lower-extendability in a population-monotonic way is a weaker requirement than *local upper-extendability*: indeed there are economies in which all envy-free allocations are *locally lower-extendable in a population-monotonic way* (Alkan, 1994).

10.6 Consistency

We now turn to *consistency* and related properties. The concept was first encountered in Subsection 8.1 in the context of classical economies. Let $\mathcal{A}$ be the family of all finite subsets of a set of "potential" objects, with generic element denoted A. Given $\sigma\colon N \to A$ and $N' \subset N$, let $\sigma(N') \equiv \cup_{N'}\sigma(i)$:

Consistency: For each $N \in \mathcal{N}$, each $(R, M, A) \in \mathcal{E}^N_{ind}$, each $z \equiv (m, \sigma) \in \varphi(R, M, A)$, and each $N' \subset N$, we have $z_{N'} \in \varphi((R'_i)_{i\in N'}, \sum_{N'} m_i, \sigma(N'))$, where $z_{N'}$ is the restriction of z to the group N', and for each $i \in N'$, R'_i is the restriction of R_i to $\mathbb{R} \times \sigma(N')$.

It is clear that both the Pareto solution and the no-envy solution are *consistent.* Are there *consistent* subsolutions of the no-envy solution?

In the one-object case, the solution F^* is a *consistent* selection from the no-envy solution. Moreover, subject to *neutrality*, it is the smallest one, as follows directly from the following theorem:

Theorem 10.9 (Tadenuma and Thomson, 1993) Domain: one infinitely divisible good; a set A of at most one object; preferences, defined on $\mathbb{R} \times A$, are monotonic with respect to money and satisfy the compensation assumption. *If a subsolution of the no-envy solution is neutral and consistent, then it contains the solution F^*.*

As a corollary of Theorem 10.9, one easily obtains a complete characterization of all solutions satisfying its hypotheses.

The results in the multiple-object case are quite different:

[46]A version of (a) holds that accommodates objects that are not desirable. The uniqueness result stated in (c) does not persist if variations in populations are considered.

Theorem 10.10 (Tadenuma and Thomson, 1991) Domain: one infinitely divisible good; a set A of objects; equal number of agents; preferences, defined on $\mathbb{R} \times A$, are monotonic with respect to money and satisfy the compensation assumption. *If a subsolution of the no-envy solution is neutral and consistent, then in fact it coincides with the no-envy solution.*

This result is also true when the objects are identical. On the other hand, and here too, whether or not the objects are identical, there are many subsolutions of the no-envy solution satisfying the weakening of *consistency* obtained by applying it only to two-agent subgroups (*bilateral consistency*), but all such solutions coincide with the no-envy solution in the two-agent case.

The next axiom pertains to the arrival of new agents, and it is the counterpart of an axiom of the same name that we encountered first in our study of the classical model (Subsection 8.1):

Converse consistency: For each $N \in \mathcal{N}$ with $|N| > 3$, each $e \equiv (R, M, A) \in \mathcal{E}^N_{ind}$, each $z \equiv (m, \sigma) \in Z(e)$, if for each $N' \subset N$ such that $|N'| = 2$, $z_{N'} \in \varphi((R_{i|\mathbb{R}\times\sigma(N')})_{i\in N'}, \sum_{N'} m_i, \sigma(N'))$, then $z \in \varphi(e)$.

Clearly, the no-envy solution is *conversely consistent*, but many proper subsolutions of it are too (as well as *neutral*). On the other hand, the Pareto solution is not, unless the objects are identical. However, we have:

Theorem 10.11 (Tadenuma and Thomson, 1991) Domain: one infinitely divisible good; a set A of objects; equal number of agents; preferences, defined on $\mathbb{R} \times A$, are monotonic with respect to money and satisfy the compensation assumption. *If a subsolution of the no-envy solution is neutral, bilaterally consistent, and conversely consistent, then in fact it coincides with the no-envy solution.*

The identical-preferences lower bound solution is *conversely consistent* but not *consistent*. The *minimal consistent enlargement* (Section 8.2) of its intersection with the Pareto solution is the Pareto solution itself. This is true when there is at most one object, when there are multiple identical objects, and when there are multiple and possibly different objects. The maximal *consistent* subsolution of the identical-preferences lower bound and Pareto solution is the no-envy solution (Bevia, 1996a)

10.7 Related models

10.7.1 Several objects per agent.

A generalization of the model is when each agent can consume several objects (in addition to the infinitely divisible good). If not otherwise indicated, all of the results below are due to Bevia (1998). The lesson that emerges from this work is that the situation is quite different from what it is in the one-object-per-person case, unless severe additional restrictions are imposed on preferences. It is true that, under similar assumptions as in the one-object-per-agent case, efficient allocations still exist and that so do envy-free allocations (this is also shown by Tadenuma, 1996). Moreover, when preferences are quasi-linear, allocations that are both envy-free and efficient exist too. If consumptions of money are required to be non-negative, existence holds if and only if the social endowment of money is at least as large as a certain amount that can be identified. This amount depends on preferences. An algorithm is available that leads to envy-free allocations (Haake, Raith, and Su, 2002). At each step, it focuses on a pair of agents between whom envy is maximal, as measured in terms of the amount of money that should be added to the bundle of the envious agent so as to make him non-envious, and adjustments are carried out so as to decrease this maximal envy.

If preferences are not quasi-linear, and even when consumptions of money are unbounded below (or the social endowment of money is sufficiently large), and preferences satisfy the compensation assumption, envy-free and efficient allocations may not exist (Tadenuma, 1996; Meertens, Potters, and Reijnierse, 2002).

Even if preferences are quasi-linear, no-envy does not imply efficiency any more. Thus, and since by definition, group no-envy still implies efficiency, the group no-envy solution may be a proper subsolution of the no-envy solution. There may be no allocation meeting the identical-preferences lower bound, although a necessary and sufficient condition on preferences can be stated guaranteeing existence. The no-envy and identical-preferences lower bound solutions are not related by inclusion. An equal-income Walrasian allocation obviously remains envy-free—in fact, it remains group envy-free—but a group envy-free allocation may not be an equal-income Walrasian allocation. Equal-income Walrasian allocations exist if preferences have additive numerical representations, a case discussed in the next paragraph. Egalitarian-equivalence applies to this model with no difficulty and existence is guaranteed very generally.

Preferences that have additive representations have been the object of particular attention. For that case, a rule is proposed by Knaster (this attribution is by Steinhaus, 1948; also, Kuhn, 1967). It consists in first assigning all objects efficiently (this is a meaningful objective because of quasi-linearity), and, using our earlier terminology, assigning consumptions of money so that all agents receive equal amounts of it above their identical-preferences lower bounds. Steinhaus also defines an asymmetric generalization of the solution. An alternative is the selection from the egalitarian-equivalence and Pareto solution obtained by choosing the null object as reference object. Interestingly, this second solution is a selection from the no-envy solution (Willson, 2003), showing that for additive preferences, no-envy is compatible with egalitarian-equivalence. Each is *money-monotonic* and each satisfies a form of *object-monotonicity*. Knaster's solution is advocated by Samuelson (1980).

Next, we turn to the implications of relational fairness requirements of monotonicity and consistency. If the social endowment of money is non-negative and all objects are desirable, the natural form of *population-monotonicity* is that upon the arrival of additional agents, each of the agents initially present should end up at most as well off as he was initially. Then, and even if preferences are quasi-linear and no other fairness requirement is imposed, no selection from the Pareto solution is *population-monotonic* (Bevia, 1996b). On the other hand, suppose that preferences are further restricted by the requirement that the free-access game associated with each economy satisfies the substitutability assumption described before Proposition 7.1. Then, the Shapley value, when applied for each economy to the free-access game associated with it, induces a rule that satisfies the property (Bevia, 1998). Much is known about *consistency*. In contrast to the one-object-per-person case, there are *consistent* subsolutions of the no-envy and Pareto solution, and *converse consistency* becomes a much stronger requirement. Nevertheless, characterizations in the spirit of Theorems 10.10 and 10.11 hold under an additional invariance requirement on solutions (Bevia, 1998).

10.7.2 Lotteries.

Population-monotonicity of rules based on lotteries is examined by Ehlers and Klaus (2001). [They also consider strategic issues. Lotteries are allowed by Crawford and Heller (1979), but mainly in the context of the strategic analysis of Divide-and-Choose.]

10.7.3 When monetary compensations are not possible

One object per agent. Economies in which money is not available to make compensations have recently been much studied, mainly in situations where preferences over objects are strict. [This literature often considers the implications of fairness conditions in conjunction with the strong requirement of immunity to strategic behavior called *strategy-proofness*, which says that no agent should ever benefit from misrepresenting his preferences.]

It is clear that punctual requirements of fairness such as no-envy and egalitarian-equivalence are not achievable here (think of situations where all agents have the same preferences), and not much can be said about these requirements. However, most of our relational requirements remain meaningful. The main lesson of this literature is that they can be satisfied, but in a rather limited way.

Of course, they are satisfied by the **single-order priority rules** defined as follows. To each order on the set of agents is associated the rule that selects, for each preference profile, the allocation at which the agent ranked first receives his most preferred object, the agent ranked second receives his most preferred object among the remaining objects, and so on. These rules, which are the counterparts for this model of what are usually called "dictatorial rules" or "sequential dictatorial rules", are a little less distasteful than they are in the classical model since here, there is a natural constraint on what an agent consumes. Even if an agent receives his most preferred object, other objects remain that are available for the other agents, and when preferences differ sufficiently, some of them may in fact receive their most preferred objects too.

Being first in an order amounts to being given ownership rights over the entire social endowment of objects. Once the agent ranked first has exercised his rights by choosing his most preferred object, the agent ranked second is given ownership rights over the remaining objects, and exercises his rights by choosing his most preferred object among them, and so on. More generally, one can "spread" priority over agents. For each object, let us specify a priority order over agents. Each agent identifies his best object over all objects. Each agent i gets his best object if he owns it. If agent i owns the best object of the owner of his own best object, they then exchange these objects and leave. Suppose there is a cycle starting with agent i such that agent i's best object is owned by the next agent in the cycle, the next agent's best object is owned by the following agent, and so on, until the last agent's best object is owned by agent i. Then these agents trade along the cycle and leave. Each object in the endowment of an agent who leaves is inherited by the first agent in the priority order for that object who does not leave. These "bequests" define revised ownership rights for the remaining agents and the process is repeated with them. Let us call it a **multiple-order priority rule**. The collection of the orders indexed by objects (the priority profile), presented as a table, is the **inheritance table** of the rule, which can also be referred to as the **hierarchical exchange rule associated with the inheritance table** (Pápai, 2000).

Although envy cannot be avoided in this model, there is a natural way to attempt to limit it. Given a single priority order on the set of agents and an allocation, say that agent i's envy of agent j is "justified" if agent i prefers the object agent j receives to the one he receives, and agent i has a higher priority than agent j (Svensson, 1994a). More generally, when the priority order may depend on the object, and given an allocation, say that agent i's envy of agent j is "justified" if agent i prefers the object agent j receives to the one he receives and he has a higher priority than agent j for that object. **A rule respects a priority profile** if it always selects an allocation at which there is no justified envy (Balinski and Sönmez, 1999) [Svensson, 1994a, addresses the issue of *strategy-proofness* when agents may be indifferent between objects.]

We also need the notion of an **acyclic priority profile** (one usually speaks of the acyclicity of a single relation, but here, the notion applies to a profile of relations). It is defined as follows: if for some object, agent i is ranked above agent j and agent j is ranked above agent k, there is no other object for which agent k is ranked above agent i (Ergin, 2002). It turns out that this is equivalent to saying that an ordered partition of the set of agents into singletons and pairs exists such that each agent in a singleton set appears in an entire row of the inheritance table, and the two agents in each pair appear only in two successive rows (Ergin, 2002). By considering ordered partitions in which components would be allowed to have up to three members, then up to four, and so on, we would dilute even more the hierarchical nature of the rule and move further and further away from single-order priority rules. But for a partition with at most two agents in each component, the resulting rule is not far from being a single-order priority rule, so we say that it is **one step away from a single-order priority rule**.[47]

Unfortunately, this is as far as one can go if imposing the requirement that when the set of available objects expands, all agents should end up at least as well off as they were initially, *object-monotonicity.* When the set of objects varies, the question is whether in defining an economy, one specifies preferences over all potential objects, or only over existing objects. We choose the latter formulation, for which we can state a result that takes a very simple form:[48]

Theorem 10.12 (Ehlers and Klaus, 2003) Domain: at least as many potential objects as agents; strict preferences over objects; either one of the following holds: (a) real objects may or may not be preferred to the null object; (b) real objects are always preferred to the null object. *A selection from the Pareto solution is object-monotonic if and only if it is one step away from a single-order priority rule.*

The implications of *consistency* for this model are explored by Ergin (2000). His main theorem involves no efficiency requirement, but we state the corollary obtained by adding efficiency:

Theorem 10.13 (Ergin, 2000) Domain: a set of objects; strict preferences over individual objects. *A selection from the Pareto solution is neutral and consistent if and only if it is a single-order priority rule.*

For solution correspondences, a characterization is available based on the axioms of Theorem 10.13 together with *converse consistency.* All admissible rules are obtained as certain unions of single-order priority rules.

The following theorem states that within the class of multiple-order priority rules, either one of two important properties implies that the orders are strongly correlated:

Theorem 10.14 (Ergin, 2002) Domain: a set of objects; strict preferences over individual objects. *A multiple-order priority rule is a selection from the*

[47]These rules are usually referred to as "mixed dictator-pairwise-exchange rules", or "restricted endowment inheritance rules".

[48]Ehlers and Klaus (2003) choose the former formulation and their characterization also involves the requirement that, given two economies such that the restrictions of preferences over the set of existing objects are the same, the rule makes the same choice: the preferences over objects that are not available should not affect the choice.

Pareto solution or satisfies consistency, *if and only if it is one step away from a single-order priority rule.*

[Another requirement on a rule for which the conclusion of Theorem 10.14 applies is *group strategy-proofness*, that is, immunity to joint misrepresentation of preferences by the agents in any group. The implications for selections from the Pareto solution of *population monotonicity* and *strategy-proofness* together are described by Ehlers, Klaus, and Papai (2002). The only rules passing these tests are one step away from single-order priority rules reformulated so as to apply to this variable-population context. The order used for each population should be induced from a single "reference" order over the entire set of potential agents. The class of selections from the Pareto solution satisfying *consistency* and *strategy-proofness* is characterized by Ehlers and Klaus (2005a,b). Kesten (2003, 2004a, 2004b) are other contributions to this line of investigation.]

Several objects per agent. Still assuming that money is not available, we now imagine that agents can receive several objects. Then, their preferences are defined over sets of objects. No selection from the Pareto solution satisfies *welfare-domination under preference replacement* (Klaus and Miyagawa, 2001). [The implications of *resource-monotonicity*, *population monotonicity*, and *consistency* have also been studied for this problem but mainly when imposed in conjunction with *strategy-proofness*, so we will not elaborate (Klaus and Miyagawa, 2001; Ehlers and Klaus, 2003). These combinations of properties mainly result in some form of priority rule.]

The model is also considered by Herreiner and Puppe (2002), who propose the following fairness criterion: For each allocation and each agent, one determines the rank in his preference of the set he receives. An allocation is then evaluated by means of the highest (across agents) such rank. Thus, the focus is on the agent who is treated the worse according to these ranks. An iterative procedure can be defined that produces the efficient allocation that is best according to this criterion among all efficient allocations.[49]

Brams and Fishburn (2000) for $|N| = 2$ and Edelman and Fishburn (2001) for $|N| > 2$ examine the special case when agents have the same preferences over individual objects but possibly different preferences over sets of objects.

[49]Herreiner and Puppe (2002) also discuss variants of this criterion and identify situations (they are quite limited) under which an envy-free allocation may be obtained. See also Ramaekers (2006).

These preferences are assumed to have additive representations in the second paper and to satisfy a slightly more general property in the first paper. Envy-free allocations may not exist—in fact, the more preferences agree, the less likely such allocations are to exist—so they propose alternative criteria. These studies underscore the importance of the relation between the number of objects and the number of agents. Brams, Edelman, and Fishburn (2003) pursue this analysis without the assumption that preferences over individual objects are the same, and propose, in addition to requirements related to no-envy, some that are based on comparing the numbers of objects received by the various agents. Brams, Edelman, and Fishburn (2001) relate the no-envy condition in the context of this model to efficiency, and to the Rawlsian and Borda choice rules. For one of their examples, there is a unique envy-free allocation, but it is not efficient.

Models with individual endowments. The possibility that agents are endowed with objects is first considered by Shapley and Scarf (1974). Situations when some objects are initially individually owned and others are commonly owned, as in Subsection 8.3, are also of interest, residential housing on a university campus being an illustrative example, since current renters are usually allowed to keep the units they occupy from one year to the next. An application to kidney exchange is discussed by Roth, Sönmez, and Ünver (2004). [Abdulkadiroğlu and Sönmez, 1999, define a *strategy-proof* selection from the Pareto solution that respects individual ownership. See also Sönmez and Ünver, 2005.]

Lotteries. Rules based on lotteries are examined by Hylland and Zeckhauser (1979), Abdulkadiroğlu and Sönmez (1998, 1999), and Bogomolnaia and Moulin (2002, 2003, 2004). Demko and Hill (1988) consider the several-objects-per-agent case in this context.

10.7.4 Compensations.

Consider the problem of allocating a single infinitely divisible good, "money", among agents characterized by talent or handicap variables, which cannot be transferred. Preferences are defined over the cross-product of the real line and the set of possible values of these variables. Each agent can be understood as endowed with a particular one of these objects, but no exchange of objects can take place. How should money be divided? This model, formulated by Fleurbaey (1994, 1995b), has been analyzed from a variety of angles. (Bossert, 1995; Bossert and Fleurbaey, 1996; Fleurbaey and Maniquet, 1996b, 1998, 1999; Iturbe-Ormaetxe and Nieto, 1996a; Otsuki, 1996b; Sprumont, 1997; Maniquet, 1998; Bossert, Fleurbaey, and van de Gaer, 1999). Fleurbaey and Maniquet (2003, Chapter 21) survey this work in detail and discuss how it fits in with the literature under discussion here. So, we will not elaborate.

11 Single-peaked preferences

Consider a one-commodity model where preferences are single-peaked: up to some critical level, an increase in an agent's consumption increases his welfare but beyond that level, the opposite holds.

An example of a situation of this kind is distribution at disequilibrium prices in a two-good economy with strictly convex preferences; the restrictions of such preferences to budget lines are single-peaked. Alternatively, imagine a task that is to be divided among a group of agents who are jointly responsible for it, and suppose that each agent enjoys performing it up to a point, but that additional time spent at it decreases the agent's welfare. Then, the agent has single-peaked preferences over the time performing the task. If the task has to be completed, the feasibility requirement is that the sum of the amounts of time spent by the agents performing it should be *equal* to the total time needed for completion. Depending upon whether the sum of the preferred amounts is greater or smaller than completion time, agents may have to supply more than their preferred amounts, or they may have to supply less. The formal model is introduced by Sprumont (1991).

It turns out that a certain rule, known as the uniform rule, is central in solving this class of problems, independently of the angle from which we attack them; whether we consider changes in the amount to divide, or in the population (here, monotonicity or consistency requirements have been considered), or in the preferences of some agents. [The rule also has very desirable strategic properties, and it is in the context of a study of *strategy-proofness* that it first emerged.]

11.1 The model

Each agent $i \in N$ is equipped with a continuous preference relation R_i defined on $\mathbb{R}_+$ with the property that there is a number, denoted $p(R_i)$, and called his **peak amount**, such that for each pair $\{z_i, z_i'\} \subset \mathbb{R}_+$, if $z_i' < z_i \leq p(R_i)$ or $p(R_i) \leq z_i < z_i'$, then $z_i \; P_i \; z_i'$: the relation is **single-peaked**. Let $\mathcal{R}_{sp}$ denote the class of all such relations. Given $z_i \in \mathbb{R}_+$, let $r_i(z_i)$ be the amount on the other side of his peak amount that agent i finds indifferent to z_i, if there is such an amount. If there is no such amount and $z_i < p(R_i)$, let $r_i(z_i) \equiv \infty$, and otherwise, let $r_i(z_i) \equiv 0$. Let $\Omega \in \mathbb{R}_+$ denote the social endowment. An **economy** is a pair $e \equiv (R, \Omega) \in \mathcal{R}_{sp}^N \times \mathbb{R}_+$ and $\mathcal{E}_{sp}^N$ is the domain of all economies with agent set N. The commodity is not freely disposable. Thus, a **feasible allocation** for $e \equiv (R, \Omega) \in \mathcal{E}_{sp}^N$ is a list $z \equiv (z_i)_{i \in N} \in \mathbb{R}_+^N$ such that $\sum z_i = \Omega$. Let $\boldsymbol{Z(e)}$ denote the set of feasible allocations of e.

11.2 Basic solutions

It is easy to check that an allocation $z \in Z(e)$ is **efficient** for $e \equiv (R, \Omega) \in \mathcal{E}_{sp}^N$ if and only if (i) when $\Omega \leq \sum p(R_i)$, then for each $i \in N$, $z_i \leq p(R_i)$, and (ii) when $\Omega \geq \sum p(R_i)$, then for each $i \in N$, $z_i \geq p(R_i)$: all consumptions should be "on the same side" of the peak amounts (Sprumont, 1991). In the first case, we can say that "there is too little of the commodity" (too little because everyone has to receive at most his peak amount), and in the second case, that "there is too much" (too much because everyone has to receive at least his peak amount).

Figure 11 illustrates an **envy-free** and efficient allocation for $N \equiv \{1, \ldots, 7\}$ and $e \equiv (R, \Omega) \in \mathcal{E}_{sp}^N$ such that $\sum p(R_i) > \Omega$. In the example, agents are ordered by their peak amounts: $p(R_1) \leq \cdots \leq p(R_7)$. As we just saw, by efficiency, no one receives more than his peak amount. By no-envy, agents are partitioned into groups—in the example, they are $\{1, 2, 3\}$, $\{4\}$, $\{5\}$, and $\{6, 7\}$—with all agents in each group receiving equal amounts. For each group G, the common consumption of the members of the group receiving the next greatest amount is at least $\max_{i \in G} r_i(z_i)$.

The question of existence of envy-free and efficient allocations is given a very simple positive answer later on. In fact, the set of these allocations is usually quite large and one of our objectives is to identify well-behaved selections from the no-envy and Pareto solution.

The **equal-division lower bound** is defined as in classical economies,

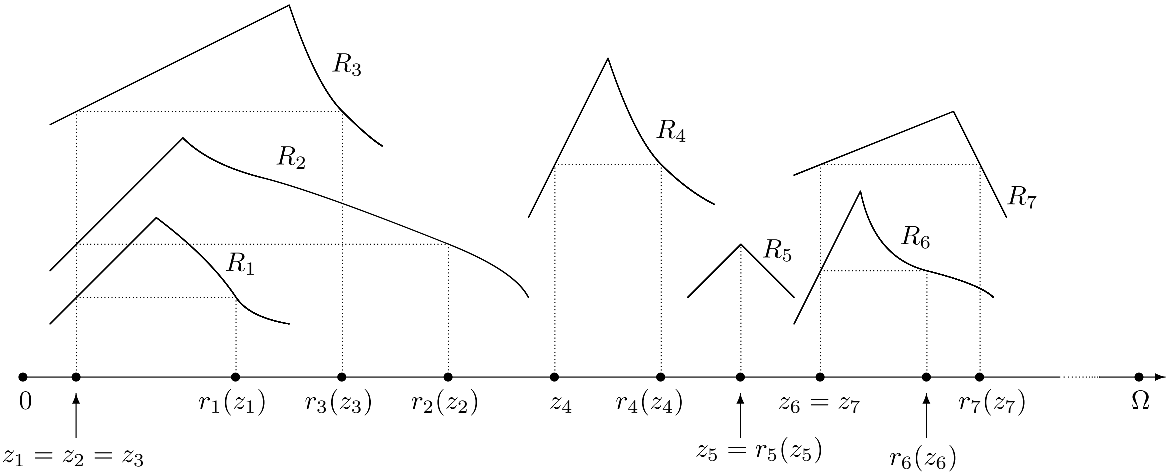


Figure 11: An envy-free and efficient allocation in an economy with single-peaked preferences. When there is not enough of the commodity, efficiency requires that each agent should receive at most his peak amount. For no-envy, no agent's consumption should fall in the interval of peak consumptions of any other agent at his consumption. All of these requirements are met here.

and so is the **equal-division core**. There always are allocations meeting the equal-division lower bound, but the equal-division core may be empty. However, if the blocking requirement for each group is strengthened by insisting that each agent in the group should be made better off by redistribution of the resources the group controls, then non-emptiness is recovered, as we will see.[50]

A number of interesting solutions can be defined for this model by taking advantage of its special features. Quite a few are *single-valued*. Here are some of them. The first one is the most frequent suggestion in the context of rationing: Given $e \equiv (R, \Omega) \in \mathcal{E}^N_{sp}$, the allocation $z \in Z(e)$ is the **proportional allocation of *e*** if, when at least one peak amount is positive, there is $\lambda \in \mathbb{R}_+$ such that for each $i \in N$, $z_i = \lambda p(R_i)$. If all peak amounts are zero, $z \equiv (\frac{\Omega}{|N|}, \ldots, \frac{\Omega}{|N|})$. It is its **equal-distance allocation** if there is $\lambda \in \mathbb{R}_+$ such that for each $i \in N$, $z_i = \max\{p(R_i) - \lambda, 0\}$. It is its **equal-preferred-sets allocation** if (i) when $\Omega \leq \sum p(R_i)$, then for each $\{i, j\} \subseteq N$, $z_i - r_i(z_i) = z_j - r_j(z_j)$, or $z_i = 0$, and (ii) when $\Omega \geq \sum p(R_i)$, then for each $\{i, j\} \subseteq N$, $z_i - r_i(z_i) = z_j - r_j(z_j)$.

The following rule will turn out to be most central in our analysis:

Definition (Bénassy, 1982) Given $e \equiv (R, \Omega) \in \mathcal{E}^N_{sp}$, the allocation $z \in Z(e)$ is the **uniform allocation of *e*** if (i) when $\Omega \leq \sum p(R_i)$, there is $\lambda \in \mathbb{R}_+$ such that for each $i \in N$, $z_i = \min\{p(R_i), \lambda\}$, (ii) when $\Omega \geq \sum p(R_i)$, there

[50] The notion of egalitarian-equivalence cannot be directly applied in this context but a reformulation is possible that produces a well-defined solution (Chun, 2000).

is $\lambda \in \mathbb{R}_+$ such that for each $i \in N$, $z_i = \max\{p(R_i), \lambda\}$.

The proportional, equal-distance, and equal-preferred-sets rules are motivated by the desire to distribute "evenly" across agents deficits or surpluses (the difference $\Omega - \sum p(R_i)$ when it is negative or positive respectively). The uniform rule does not seem to achieve this objective since it assigns to some agents their peak amounts, and it assigns to the others equal amounts (it ignores differences in the peak amounts of the members of the latter group). However, it does select an allocation that is envy-free (this solves the existence question for these allocations, as announced earlier), and it meets the equal-division lower bound, in contrast with the proportional, equal-distance, and equal-preferred-sets rules.

The uniform rule depends only on the profile of peak amounts—we say that it satisfies **peak-only**. This property is satisfied by many other solutions such as the Pareto solution, and by the equal-distance and proportional rules. However, we have the following characterization:

Theorem 11.1 (Thomson, 1994c) Domain: one good; single-peaked preferences. *The uniform rule is the only subsolution of the no-envy and Pareto solution satisfying* peak-only.

The uniform allocation of an economy is also its only efficient allocation at which each agent receives an amount that he finds at least as desirable as any convex combination of the amounts received by all agents (recall Kolm's generalization of no-envy; Chun, 2000). When there is too little of the commodity, it is also the only allocation maximizing the factor α such that each agent finds the amount he receives at least as desirable as the proportion α of the social endowment. When there is too much, replace "maximizing" by "minimizing". Finally, the uniform rule is the only selection from the Pareto solution minimizing either one of the following two alternative ways of measuring the disparity among the amounts received by the various agents: (i) the difference between the smallest amount anyone receives and the greatest amount anyone receives; (ii) the variance of the amounts they all receive (Schummer and Thomson, 1997). Parallel results can be obtained if the Pareto requirement is replaced by *peak-only* (Kesten, 2004c).

The **group no-envy** solution, defined as in the classical case, is sometimes empty but the uniform allocation satisfies the weaker requirement on an allocation that no group of agents can make all of its members better off, assuming that each of them is given access to an equal share of the social endowment, by redistributing the resources it controls in total.

11.3 Resource-monotonicity

We begin our study of relational requirements for this model by considering changes in the social endowment. Since preferences are not monotonic, it would of course make no sense to ask that when it increases, all agents should be made at least as well off as they were initially. Instead, a natural expression of the idea of solidarity is that *either* all agents should be made at least as well off as they were initially *or* that they should all be made at most as well off. In the presence of efficiency, the first case applies in particular if all agents initially receive at most as much as their peak amounts and the social endowment does not increase too much, and the second case if initially they already receive at least as much as their peak amounts. Unfortunately, none of the rules defined above satisfies this requirement, as can be seen by means of simple examples. Although it is met on a large subdomain of our primary domain by variants of the equal-preferred-sets rule, it turns out to be incompatible with no-envy as well as with the equal-division lower bound, even if efficiency is dropped. The reason is that the increase can be so disruptive that it turns an economy in which there is too little to one in which there is too much. The relevance of this distinction to the possibility of monotonicity should not be surprising, and it suggests limiting the application of the requirement to situations in which no such switches occur. We write this weaker property for *single-valued* solutions:

One-sided resource-monotonicity: For each $(R, \Omega) \in \mathcal{E}^N_{sp}$ and each $\Omega' \in \mathbb{R}_+$, if either $\Omega' \leq \Omega \leq \sum p(R_i)$ or $\sum p(R_i) \leq \Omega \leq \Omega'$, then $\varphi(R, \Omega)\ R\ \varphi(R, \Omega')$.

This property is satisfied by all the rules mentioned earlier. Yet, when imposed in conjunction with no-envy and efficiency, it essentially singles out the uniform rule. The "essentially" is a reference to a restriction on the domain: for each $i \in N$, 0 should have a finite equivalent ($r_i(0) < \infty$). If the restriction is not imposed, generalizations of the uniform rule become admissible. The class they constitute can be characterized, but we omit their formal description.

Theorem 11.2 (Thomson, 1994b) Domain: one good; single-peaked preferences for which 0 has a finite equivalent. *The uniform rule is the only selection from the no-envy and Pareto solution satisfying one-sided resource-monotonicity.*

In this characterization, the Pareto requirement can be replaced by *peak-only* (Kesten, 2004c), or by **resource-continuity**, which says that small changes in the resource should not lead to large changes in the selected allocation (Ehlers, 2002d).

11.4 Welfare-domination under preference-replacement

Next, we consider the replacement of the preferences of some agents by some other preferences, and require that the welfares of all other agents should be affected in the same direction by the replacement (Subsection 7.2). This property of **welfare-domination under preference-replacement** is also quite strong: once again, no selection from the no-envy and Pareto solution satisfies it. The reason is that here too, the change may turn an economy in which there is too little to one in which there is too much, or conversely. So, let us consider the weaker property of **one-sided welfare-domination under preference-replacement**, obtained by limiting attention to changes in preferences that do not reverse the direction of the inequality between the amount to divide and the sum of the peak amounts.

Many rules satisfy this property, including the uniform, proportional, equal-distance, and equal-preferred-sets rules, but here too, we have a uniqueness result. It is another characterization of the uniform rule, based on this property. It also involves *replication-invariance*. This requirement is very weak, being met by the Pareto solution, the no-envy solution, their intersection, the uniform rule, and the equal-division lower bound solution. Nevertheless, we have:

Theorem 11.3 (Thomson, 1997a) Domain: one good; single-peaked preferences for which 0 has a finite equivalent. *The uniform rule is the only selection from the no-envy and Pareto solution satisfying weak replication-invariance and one-sided welfare-domination under preference-replacement.*

The independence of *replication-invariance* from the other axioms in Theorem 11.3 is established by Klaus (1999).

There are selections from the equal-division lower bound and Pareto solution satisfying *welfare-domination under preference-replacement* other than the uniform rule. They constitute a convex class.

11.5 Separability

Here, we consider the requirement that, given any two economies having a group of agents in common, if the agents in this group receive the same aggregate amount in both, then each of them should receive the same amount in both (Chun, 2006c). Note that the economies that are compared may have different social endowments:

Separability: For each pair $N \subset \mathcal{N}$, each $e \equiv (R, \Omega) \in \mathcal{E}^N_{sp}$, each $N' \subset N$, and each $e' \equiv (R', \Omega') \in \mathcal{E}^{N'}_{sp}$, if $R_{N'} = R'_{N'}$ and $\sum_{N'} \varphi_i(e) = \sum_{N'} \varphi_i(e')$, then $\varphi_{N'}(e) = \varphi_{N'}(e')$.

We have the following characterizations:

Theorem 11.4 (Chun, 2006c) Domain: one good; single-peaked preferences. *The uniform rule is the only selection* (i) *from the no-envy and Pareto solution satisfying* resource-continuity *and* separability*;* (ii) *from the equal-division lower bound and Pareto solution satisfying* resource-continuity *and* separability*;*[51] (iii) *from the equal-division lower bound solution satisfying* one-sided resource monotonicity *and* separability*.*

In (i), the conclusion persists if *replication-invariance* is imposed instead of *resource continuity* (Klaus, 2006). Other characterizations of the uniform rule not involving efficiency are available (Chun, 2003). The distributional requirements are either no-envy or the equal-division lower bound and the relational requirements are *replication-invariance*, *one-sided population-monotonicity*, and *separability.* Additional properties involving simultaneous changes in several parameters can be formulated (Chun, 2006c, discusses their logical relations).

[51] We do not include *replication invariance*, which Chun had imposed, as Klaus (2006) showed that it is redundant.

11.6 Population-monotonicity

Turning now to variations of population,we first note that here too, it would make no sense to require as we did for classical economies, that the departure of some agents, resources being kept fixed, should make all of the remaining agents at least as well off as they were initially. Indeed, when there is initially too much of the commodity, this departure may permit a uniform welfare improvement for them. Therefore, we require instead that their welfares should all be affected in the same direction.

This requirement is incompatible with no-envy as well as with the equal-division lower bound. However, as was the case in our examination of the impact of changes in the social endowment, on important subdomains of our primary domain, there are selections from the Pareto solution that do satisfy it, namely, variants of the equal-preferred-sets rule.

Just as was the case there, the reason why *population-monotonicity* is so strong is that it sometimes forces comparisons between economies in which there is too little to divide and economies in which there is too much. We therefore weaken it by excluding from its coverage changes in populations that reverse the direction of the inequality between the amount to divide and the sum of the peak amounts, thereby obtaining **one-sided population-monotonicity**. We have the following uniqueness result:

Theorem 11.5 (Thomson, 1995a) Domain: one good; single-peaked preferences for which 0 has a finite equivalent. *The uniform rule is the only selection from the no-envy and Pareto solution satisfying* replication-invariance *and* one-sided population-monotonicity*.*

The independence of *replication-invariance* from the other axioms in Theorem 11.5 is established by Klaus (1999). Also, the Pareto requirement can be replaced by *peak-only* without affecting the conclusion (Kesten, 2004c).

11.7 Consistency

Recall that a (possibly multi-valued) solution is **consistent** (Section 8) if the desirability of an allocation it selects for some economy is not affected by the departure of some agents with their assigned consumptions:

Consistency: For each $N \in \mathcal{N}$, each $(R, \Omega) \in \mathcal{E}^N_{sp}$, each $z \in \varphi(R, \Omega)$, and each $N' \subset N$, we have $z_{N'} \in \varphi(R_{N'}, \sum_{N'} z_i)$.

Quite a few solutions are *consistent*, including the Pareto solution, the no-envy solution, and the uniform and proportional rules. The equal-division lower bound solution is not. However, *consistency*, together with the mild and self-explanatory requirement of **resource upper hemi-continuity**, when imposed on a subsolution of the no-envy and Pareto solution, implies that this solution contains the uniform rule:

Theorem 11.6 (Thomson, 1994c) Domain: one good; single-peaked preferences. *If a subsolution of the no-envy and Pareto solution satisfies* resource upper hemi-continuity *and* consistency, *then it contains the uniform rule.*

From this theorem can easily be derived a complete characterization of the class of solutions satisfying the hypotheses. A corollary of that characterization is that if the solution is in addition required to be *single-valued*, then in fact, it coincides with the uniform rule. A similar result holds when the equal-division lower bound is imposed instead of no-envy.

A direct proof of this corollary, exploiting more completely the *single-valuedness* requirement, is possible (Dagan, 1996). The main step is showing that if a *single-valued* selection from the no-envy and Pareto solution is *consistent*, then it satisfies *peak-only*. The conclusion follows then by Theorem 11.1. This characterization involves no continuity requirement.

The uniform rule is also the only *single-valued* selection from the equal-division lower bound and Pareto solution to be *replication-invariant* and *consistent* (a counterpart of Theorem 8.3), and to be *anonymous* and *conversely consistent* (see Section 8 for a formulation of this property, whose adaptation to the current model is straightforward) (Thomson, 1994c).

When the set of agents is modelled as a non-atomic measure space, the uniform rule is the only subsolution of the equal-division lower bound and Pareto solution to be *consistent* (This is an application of a result due to Thomson and Zhou, 1993, mentioned in Section 8.1).

Additional characterizations of the uniform rule are available (Kesten, 2004c). They involve various combinations of the properties just seen. Some rely on *peak-only*, and some dispense with efficiency.

11.8 Extensions and related models

11.8.1 Other characterizations of the uniform rule

Sönmez (1994) strengthens *one-sided resource-monotonicity*—let us call the stronger requirement *strong one-sided resource-monotonicity*—and characterizes the uniform rule as the only selection from the equal-division lower bound solution to satisfy this property and to be *consistent*. For a parallel strengthening of *one-sided population-monotonicity*, he characterizes the rule as the only selection from the equal-division lower-bound solution to satisfy in addition *replication-invariance* and *consistency* (or *converse consistency*). He does not impose efficiency. It turns out that in fact, *one-sided resource-monotonicity* could have been imposed instead. When consumption spaces are bounded above, *strong one-sided resource-monotonicity* by itself implies efficiency (Ehlers, 2002a). Uniqueness still holds with *bilateral consistency* instead of *consistency* Kesten (2004c).

In this model, the idea of resource monotonicity can also be formulated in physical terms: when the amount to divide increases, each agent should receive at least as much as he did initially (Otten, Peters, and Volij, 1996, consider variants of the idea; Moulin, 1999). Otten, Peters, and Volij (1996) require that two agents with the same preferences should receive amounts that are indifferent according to these common preferences, and impose a monotonicity property with respect to simultaneous changes in the social endowment and preferences. Their main result is a characterization of the uniform rule.

It is easy to see that in the presence of efficiency, the various notions of *resource-monotonicity* that have been proposed for the model are equivalent (Ehlers, 2002a). The implications of several combinations of solidarity requirements with respect to changes in several of the parameters of the problem, as well as *consistency*, are explored by Chun (2003). Additional results, which are also characterizations of the uniform rule, involve no efficiency. [For the variant of the model in which consumption spaces are bounded above, a family of "fixed path" rules that can be seen as generalizations of the uniform rule emerge as the only selections from the Pareto solution to satisfy *resource-monotonicity* expressed in physical terms, *consistency*, and *strategy-proofness*; Moulin, 1999. Ehlers, 2002a, 2002b, establishes a variety of related results, some not involving efficiency.]

11.8.2 Economies with individual endowments and economies with a social endowment and individual endowments

All of the rules we have discussed can be easily extended to the variant of the model obtained by introducing individual endowments. Various fairness issues for this model are considered by Moreno (2002), Klaus (1997a, 2001b), and Klaus, Peters, and Storcken (1997a). [*Strategy-proofness* is studied by Klaus, Peters, and Storcken, 1997a,b, and Barberà, Jackson, and Neme, 1997, the latter invoking it in conjunction with *welfare-domination under preference-replacement*.]

Even more general are situations where, in addition to individual endowments, we specify an amount interpreted as a collective obligation to or from the outside world (recall the "generalized" economies of Subsection 8.3). Various ways of generalizing the punctual fairness requirements that have been central in this survey, and issues of *monotonicity*, with respect to the individual endowments, the collective obligation, in addition to *consistency* and *population-monotonicity*, have been addressed (Thomson, 1996; Herrero, 2002).

In these studies, a rule that is the natural extension of the uniform rule has most frequently emerged. Like the uniform rule, it gives all agents equal opportunities, but this time for change in consumptions. For each $i \in N$, let $\omega_i \in \mathbb{R}_+$ be agent i's endowment, and let $T \in \mathbb{R}$ be the collective obligation. Assume that $\sum \omega_i + T \geq 0$, since otherwise there would be no feasible allocation. Now, given $\lambda \in \mathbb{R}$, if $\sum \omega_i + T \leq \sum p(R_i)$, give to each agent $i \in N$ the maximizer of his preferences among all amounts at most as large as $\omega_i + \lambda$ if this set is non-empty. Give him 0 otherwise. If $\sum \omega_i + T \geq \sum p(R_i)$, give to each agent $i \in N$ the maximizer of his preferences among all amounts at least as large than $\omega_i + \lambda$. In each case, choose λ so that the list of maximizers defines a feasible allocation. (To apply the definition to economies in which there is no collective obligation, just set $T = 0$.)

11.8.3 Multi-commodity generalization

A multi-commodity version of the single-peaked assumption is easily defined. For such a model, a generalization of the "equal-slacks Walrasian solution" (Mas-Colell, 1982) is axiomatized along the lines of Schummer and Thomson's (1997) axiomatization of the uniform rule (Amoros, 1999). [Amoros, 2002, defines an extension of the uniform rule that remains a *strategy-proof* selection from the no-envy solution, but it only satisfies a weak form of efficiency. Sasaki, 2003, also focuses on *strategy-proofness*.]

11.8.4 Lotteries

A probabilistic version of the uniform rule is defined and characterized by Sasaki (1997). [Probabilistic rules are studied for a version of the model in which the dividend comes in discrete units by Ehlers and Klaus, 2003, who focus on *strategy-proofness*. Their main result is a characterization of a probabilistic version of the uniform rule.]

11.8.5 An application to a pollution control problem

The problem of allocating pollution permits can be seen as a variant of the present model, and characterizations of a counterpart of the uniform rule for it have been developed (Kıbrıs, 2003).

11.8.6 Single-troughed preferences

Concerning the dual case of single-troughed preferences, not much can be said about fairness. Preferences not being convex, there may be no efficient allocation that is also envy-free or meets the equal-division lower bound. [The model has mainly been studied in the context of strategic issues, by Klaus, Peters, and Storcken, 1997b; Klaus, 2001a; Ehlers, 2002c.]

12 Non-homogeneous continuum

Here, we consider another non-classical problem, that of dividing a heterogeneous commodity, such as land or time. In such situations, equal division has no economic meaning, even when it can be defined in physical terms (length or surface area, say). However, our central criteria (no-envy; egalitarian-equivalence) remain applicable. A large literature concerns the case when preferences can be represented by atomless measures, and additional criteria can be formulated for that case. Representability by measures means that the value to an agent of each "parcel" (measurable subset of the dividend) is independent of which other parcels he may already consume. It precludes complementarity or substitutability between parcels.

This literature has addressed several issues: (i) The existence of partitions satisfying various criteria of fairness as well as certain topological and geometric requirements, for instance whether each agent's component of the partition is connected; whether it is connected to a preexisting parcel that the agent already consumes (and that is added to the specification of the problem); when the dividend is a subset of a Euclidean space, whether the boundaries between components of the partitions are hyperplanes, or perhaps parallel hyperplanes. (ii) The construction of iterative procedures leading to such partitions, the distinction being made between continuous procedures ("moving knife" procedures in the terminology of Brams and Taylor, 1995, or "moving knives" procedures, as there may be more than one knife moving at once), or discrete; the number of steps required, and whether the fairness criterion ends up being met exactly or only in some approximate sense. [The strategic properties of the procedures, that is, whether, when agents behave strategically, it produces the desired partitions, are also important. Here, the answer depends on which behavioral assumptions are made, but most writers have assumed that agents follow a maximin criterion. The implementation literature, which we do not review here, has focused on Nash and related behaviors.] The existential part of this program often relies on tools of measure theory, and mathematicians have been the main contributors.

If no restrictions are imposed on preferences apart from continuity and monotonicity (of two parcels related by inclusion, the larger one is preferred to the smaller one), it is easy to show that envy-free and efficient partitions may not exist (Berliant, Dunz, and Thomson, 1992). Indeed, there is a sense in which economies with homogeneous goods but non-convex preferences can be seen as a special case of the economies considered here, and for these economies, there may be no such partition (Subsection 4.3). However, if preferences can be represented by measures, envy-free partitions exist:

Theorem 12.1 (Weller, 1985) Domain: measurable space; preferences representable by finite and atomless measures. *Envy-free and efficient partitions exist.*

An existence result that pertains to the notion of a group envy-free partition is available:

Theorem 12.2 (Berliant, Dunz and Thomson, 1992) Domain: measurable subset of a finite-dimensional Euclidean space; strictly monotonic preferences that are representable by atomless measures. *Group envy-free partitions exist.*

An interesting special case is the one-dimensional case when the continuum has to be divided into intervals, each agent receiving one. It has many applications: division of an interval of time, a length of road, etc. The existence of envy-free partitions when preferences are represented by measures is proved by Woodall (1980). A more general existence result holds however. It only relies on continuity of preferences and a very weak monotonicity assumption:

Theorem 12.3 (Stromquist, 1980) Domain: interval in $\mathbb{R}_+$; preferences, defined over intervals, are such that each interval is at least as desirable as the empty set. *Envy-free partitions into intervals exist.*

An even stronger result holds for preferences that may exhibit a certain form of consumption externalities, no-envy being appropriately reformulated. The proof is existential. Under essentially the same assumptions, Su (1999) gives an algorithmic proof for the existence of partitions satisfying no-envy, not exactly, but up to any degree of approximation.

If the one-dimensional continuum is a closed curve, there may be no envy-free partition (Thomson, 2007) unless $|N| = 2$, in which case existence holds as soon as the preferences of at least one agent are representable by a measure.

Under monotonicity of preferences, no-envy implies efficiency (Berliant, Dunz, and Thomson, 1992). This implication is reminiscent of an earlier result pertaining to the assignment of objects when monetary compensations are possible (Theorem 10.1). These authors describe a class of models for which no-envy implies efficiency, and in fact group no-envy.

As far as egalitarian-equivalent and efficient allocations are concerned, existence is guaranteed under continuity and strict monotonicity of preferences (Berliant, Dunz and Thomson, 1992).

An extensive literature pertains to the case when preferences can be represented by countably additive and atomless measures,[52] and the requirement is that for each agent, the value to him of his assignment should be at least $\frac{1}{n}$ times his value of the dividend. Let us refer to it as the $\boldsymbol{\frac{1}{n}}$**–lower-bound**.

[52]Properties of preferences guaranteeing representability by such measures are given by Barbanel and Taylor (1995).

It is the form taken by the identical-preferences lower bound for this model. Some of the early literature searched for partitions such that for each agent, the value to him of his assignment should be equal to $\frac{1}{n}$ times his value of the dividend (which is of course in violation of efficiency). A more demanding requirement than the $\frac{1}{n}$–lower-bound is that in addition and when possible, the inequality be strict for each agent. Given a list $\alpha \in \Delta^N$ of "shares", chosen so as to reflect the relative importance to be given to the various agents—asymmetric treatment may be important in some circumstances—the notion can be modified by requiring that each agent i should receive at least α_i times his value of the dividend. Let us refer to it as the **$\boldsymbol{\alpha}$–lower-bound**. The existence of partitions satisfying these notions and generalizations is an easy implication of the Dvoretsky, Wald, and Wolfovitz (1951) theorem (Barbanel and Zwicker, 1995). Barbanel (1996b) proposes yet more general criteria. No-envy remains of course applicable for this model and we have already cited Weller 's (1985) existence result. Until relatively recently and except for that paper, efficiency issues had been ignored in this literature—this is the case in the papers just cited—computational and algorithmic aspects of the problem being given a central place instead.

The existence of partitions meeting the α–lower-bound, for each $\alpha \in \Delta^N$, is in fact guaranteed under a more general assumption than in Theorem 12.1, namely that preferences be representable by an integrable function h with two arguments, the points of the dividend, and measurable subsets of it, and such that for each pair $\{B, B'\}$ of such subsets with $B' \subset B$, and each $x \in B \setminus B'$, we have $h(x, B') \geq h(x, B)$ (Berliant, Dunz and Thomson, 1992; related sufficient conditions for existence are stated by these authors). Efficiency is obtained in some approximate sense. An existence result for preferences representable by atomless concave capacities is given by Maccheroni and Marinacci (2003). Akin (1995) also goes beyond Theorem 12.1 by proving the existence of envy-free partitions for a more general notion of a partition (where agents receive "fractional" consumptions of each point of the dividend).

A succession of attempts at generalizing to more than two agents the classical two-person Divide-and-Choose scheme (one agent divides and the other chooses one of the two pieces; the divider receives the other), have been made over the years that generate partitions that are either envy-free or meet the $\frac{1}{n}$–lower-bound, two properties that the scheme satisfies (as noted earlier, some authors have required that each agent's own measure of his share be exactly $\frac{1}{n}$ of his measure of the dividend) (Knaster, 1946; Steinhaus, 1948, 1949; Dubins and Spanier, 1961; Singer, 1961; Kuhn, 1967; Austin, 1982; Sobel, 1981; Woodall, 1986). Some of these papers also cover the case when the monotonicity assumption on preferences is reversed (of two parcels related by inclusion, the smaller one is at least as desirable as the larger one), which describes situations when the dividend is a "bad". Several of these procedures yield partitions satisfying the $\frac{1}{n}$–lower-bound. It took many years until an algorithm that produces an envy-free partition in the n-person case, for arbitrary n, was discovered (Brams and Taylor, 1995). In none of these algorithmic papers is efficiency necessarily attained.

Other contributors are Hill (1983), Legut (1985, 1990), Barbanel (1995, 1996a,b), Brams, Taylor, and Zwicker (1997), Reijnierse and Potters (1998), Ichiishi and Idzik (1999), Zeng (2000), and Barbanel and Brams (2004). Brams and Taylor (1996) offer a detailed review of the literature. Robertson and Webb (1998) focus on algorithms and pay little attention to efficiency. On the other hand, Barbanel (2005), the most recent entry in the field, provides an in-depth analysis of the shape of the image of the set of feasible partitions in a Euclidean space of dimension equal to the number of agents, using their measures as representations of their preferences. It offers characterizations of its subset of efficient points. It also gives existence results for efficient and envy-free partitions.

13 Other domains and issues

We began this survey by specifying its scope as being limited to resource allocation in concretely specified economic models. We conclude by tying it to literatures concerning other models.

• Arrovian model of extended sympathy. The no-envy concept has been studied in this context (Goldman and Sussangkarn, 1978; Suzumura, 1981a, 1981b, 1983; Denicoló, 1999).

• Rights assignments. Here too, the no-envy concept has been the object of several studies (Austen-Smith, 1979; Suzumura, 1982)

• Quasi-linear model of social choice. This model is somewhat more structured, although physical resource constraints do not explicitly appear. A number of bounds on welfares have been defined, and relational fairness requirements investigated (Moulin, 1987c; Chun, 1986).

• Intertemporal allocation. Models of allocation across generations are usually formulated in utility space (Diamond, 1965, is a precursor; Svensson, 1980, is the closest in spirit to the literature we reviewed).

• Choosing a point from an interval or a closed curve when agents have single-peaked preferences. Since all agents consume the same thing, punctual fairness requirements provide little help here, but relational requirements of monotonicity are still meaningful (Thomson, 1993; Ching and Thomson, 1994; Ehlers and Klaus, 2006, Gordon, 2004). [Moulin, 1980, is the classic reference for *strategy-proofness*.]

• Strategic issues (the implementability of solutions, in particular *strategy-proofness*) have been the object of a considerable literature, most of which is reviewed in Chapter 5. A number of authors have considered implementation in the special context of fairness (Crawford, 1979; Demange, 1984; van Damme, 1986, 1992; Maniquet, 1994, 2002; Thomson, 2005).

• Cost sharing. This literature is reviewed in Chapter 6. (Moulin, and various coauthors)

• Queueing, scheduling, and sequencing. This is a very new literature (Crès and Moulin, 2001; Maniquet, 2003; Chun, 2006a,b).

14 Conclusion

When normative issues are being addressed, the likelihood of a resolution that satisfies everyone is even more remote than when only issues of efficiency are at stake. In addition, several approaches to the problem of fair allocation can be taken and we have deliberately followed only an ordinal approach, without attempting to survey the literature based on utility information. Our objective was simply to see how far this approach could take us and to explore its potential and limitations.

We now have available a well-developed theory of fair allocation that is unified in its conceptual apparatus, and well integrated with theoretical developments that have taken place in other areas of economic theory, such as the theory of mechanism design and implementation.

At this point, it might therefore be appropriate to attempt some preliminary assessment, and we close with a few general remarks intended to highlight what we think are particularly important developments.

Punctual fairness requirements. Although we have encountered a great variety of concepts, they can all be roughly divided into criteria compatible with, or in the spirit of, no-envy, and a class of "egalitarian" criteria, which we have discussed under the name of egalitarian-equivalence. In each family, noteworthy solutions have been identified. The most prominent member of the first family is the equal-division Walrasian solution. For the second family, various selections obtained by imposing some restriction on the reference bundle have proved most useful.

Relational requirements. As far as these types of axioms are concerned, and in spite of their apparent diversity, we should return to the notion of solidarity to emphasize it as a major unifying theme. Its general expression is that when a parameter entering the description of a problem changes (simultaneous changes in several parameters can be considered too), the welfares of what we call the "relevant" agents should be affected in the same direction. When efficiency is imposed as well, it is often known what that direction has to be. For instance, in a private good economy with monotonic preferences, an increase in the social endowment is required to benefit everyone (the scope of the axiom covers everyone), a requirement that we have studied under the name of *resource monotonicity.* When some agents leave, the relevant agents are the agents who stay. Together with efficiency, we derive our condition of *population monotonicity.* Sometimes it is not clear whether the change permits a Pareto improvement or whether it forces a Pareto deterioration, so solidarity takes the weaker form of a uniform change in the direction of the welfares of the relevant agents. This is the case in a private good economy when the amount available of some commodity goes up and that of another commodity goes down (once again, everyone is a relevant agent), or when the preferences of some agents change (here, the relevant agents are all the agents whose preferences are fixed). The requirement of *consistency* can also be seen in that light. This time, we imagine some agents leaving, not empty-handed but with the bundles assigned to them by the rule. Together with efficiency, the solidarity idea tells us that for the remaining agents, the rule should choose the same bundles as initially, or a Pareto-indifferent distribution of these bundles.

Solutions. Given the prominence of the Walrasian solution throughout modern economic theory, we should emphasize that the equal-division Walrasian solution has *a priori* no greater intrinsic merit than others do. Yet, the recommendation is often made to use this solution. Explicit reasons are rarely given, although the fact that it treats agents anonymously seems to be an important underlying motivation. However, most solutions commonly discussed also have this property when operated from equal division, and much more is needed to justify focusing on the Walrasian solution.

One lesson to be drawn from the work we have reviewed is that the solution does have a number of remarkable features that indeed justify that special attention be paid to it. First, as noted earlier, it is compatible with no-envy, but as opposed to this concept itself, which is satisfied by a continuum of allocations, Walrasian allocations typically are few in number; it often makes much more precise recommendations. Equal-division Walrasian allocations satisfy various notions related to no-envy, as well as criteria, adapted from no-envy, designed to deal with the fair treatment of groups. In large economies in which the set of agents is modelled as a continuum, if preferences are smooth and sufficiently dispersed, any envy-free and efficient allocation is in fact an equal-division Walrasian. Moreover, together with the concept of a Walrasian trade, that of an equal-division Walrasian allocation can form the basis for a complete and coherent theory of fair allocation, free of the conceptual difficulties encountered with some of the other solutions. When more general formulations are examined, in particular, in situations where it is deemed desirable to give agents equal opportunities to choose from a common choice set, we found that requiring the choice set to have a straight-line boundary, that is, to be a Walrasian choice set, has a number of advantages. In a model with variable populations, the equal-division Walrasian solution is the only solution to satisfy a number of appealing consistency requirements. In economies with indivisible goods, it coincides with the no-envy solution. In economies with single-peaked preferences, the special form it takes, which is known under the name of the uniform rule, was also found to be best-behaved from a number of viewpoints. [The equal-division Walrasian solution plays a prominent role in the analysis of incentive and informational issues.] These findings contribute to creating an overall picture in which the star solution is the equal-division Walrasian solution.

On the negative side, we should recognize that the equal-division Walrasian solution fails a number of important relational tests of solidarity. In particular, it fails all monotonicity tests with respect to any of the data describing the economy (resources, technology, number of agents). Moreover, better-behaved solutions exist. These solutions are various selections from egalitarian-equivalence, as well as some generalizations. If increases in the social endowment may hurt some agents when the Walrasian solution is operated from equal division, selections from the egalitarian-equivalence and Pareto solution exist that behave well in response to such changes. Similarly, in public good economies, selections from the correspondence exist that respond well to improvements in the technology, whereas in private good economies with increasing returns, an extension of the concept is shown to provide the only way to select efficient allocations that satisfy a natural individual rationality condition: this solution is such that each agent finds his bundle indifferent to the best bundle he could reach if given access to a constant-returns-to-scale economy, this reference technology being the same for all.

It would serve no purpose at this point to commit oneself to a particular concept. Certainly the recent literature has shown that no concept uniformly dominates the other, but some reassessment of the existing concepts has certainly taken place, and at the same time a variety of new concepts have been proposed and exciting results established.

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