Optimal Delegation with Multi-Dimensional Decisions *

Frédéric Koessler † David Martimort ‡

September 11, 2009

Abstract

This paper investigates optimal communication mechanisms in a framework with a two-dimensional policy space and no monetary transfers. Contrary to the one-dimensional setting, if the conflicts of interests between the principal and the agent are different on each dimension, the optimal mechanism never exhibits any pooling and the agent's ideal policies are never chosen. Simple delegation sets that trade off inflexible rules and full discretion are no longer optimal but take more complex shapes.

Keywords: Communication; Delegation; Mechanism Design; Multi-Dimensional Decision.

JEL Classification: D82; D86.

1 Introduction

Consider an informed agent who contracts with an uninformed principal. When the principal's and the agent's interests are conflicting, the principal may want to exert some ex ante control on the agent by restricting the decision set from which the agent may pick actions. Examples of such constrained delegation abound across all fields of economics and political science. A firm's CEO controls division managers by designing capital budgeting rules and allocating decision rights among unit managers.[1] Many different aspects of the firm's decisions related to product design and quality, prices, or polluting emissions are instead scrutinized by regulators. Regulations generally take the form of prices caps, minimal quality standards and other environmental norms. Central bankers face caps on inflation rates that the monetary policy they choose might induce.[2] Governments are subject to constitutional bounds on budget deficits. Lastly, Congress Committees exert ex ante control on better informed regulatory agencies by designing various administrative procedures and rules that limit bureaucratic drift.[3]

Those examples all share the common feature that principals hardly use monetary transfers to control their agents. This limited ability to rely on transfers for incentive purposes may follow

*We thank Thomas Palfrey for helpful discussions at the early stage of this project, Wouter Dessein, Aggey Semenov, Shan Zhao and Larry Samuelson for useful comments and suggestions. The usual disclaimer applies.

†Paris School of Economics and CNRS.

‡Toulouse School of Economics and EHESS.

[1] Harris and Raviv (1996) and Alonso et al. (2008).

[2] Athey et al. (2005).

[3] McCubbins et al. (1987), Huber and Shipan (2002), Epstein and O'Halloran (1999).

from a constitutional ban as in the case of U.S. regulatory agencies which are legally prevented from targeting regulated firms with specific transfers. It might also come from the fact that agents are sometimes hardly receptive to monetary incentives like, for instance, regulatory agencies when subject to legislative control. Lastly, in some contexts, principals may not be able to credibly commit to use payments to align their objectives with those of their agents.

Following the seminal works of Holmström (1984) and Melumad and Shibano (1991), those settings are fruitfully analyzed as mechanism design problems in which the principal can commit to a decision rule but cannot use monetary transfers to implement that rule.[4] With no transfers and when actions lie in a one-dimensional set, optimal communication mechanisms look crude enough. Pooling is pervasive in such cases. Quite intuitively, the principal finds it hard to induce information revelation and align conflicting objectives when he controls only a single action of the agent. In a one-dimensional setting, an optimal mechanism balances the flexibility gains of letting the agent choose freely this action according to his own private information and the agency cost coming from the fact that the principal and the agent might have conflicting objectives.

The first major result provided by the existing literature highlights the trade-off between *rules* and *discretion* that arises in such contexts. Inflexible rules allow the principal to choose his most preferred policy although they make no use of private information. Leaving discretion to the agent allows to implement state dependent actions but those choices reflect now only the agent's preferences and not those of the principal. The second important result pushed forward by the literature is that the optimal mechanism (when continuous) can be implemented by means of simple delegation sets which put bounds on the agent's action. This is an important theoretical insight because it reduces the design of the mechanism to finding those bounds. This simplification has also a great value in view of the implementation of the optimal mechanism in practice. Finally, the one-dimensional model has also provided one important comparative statics. The trade-off between *rules* and *discretion* is tilted towards leaving discretion when the agent's ideal policy is closer to that of the principal: the so-called *"Ally Principle"*.[5]

Moving beyond the one-dimensional case raises robustness challenges for each of those results. First, one may wonder whether the trade-off between rules and discretion remains when several of the agent's activities can be controlled by the principal. Clearly screening possibilities improve and rules seem less attractive but by how much? Second, in a multi-dimensional context, the agency problem between the principal and his agent may not only be related to their average conflict of interests over all dimensions but also to the distribution of those conflicts across the different activities. These questions are highly relevant not only from a pure theoretical viewpoint as a

[4]Armstrong (1994), Baron (2000), Martimort and Semenov (2006), Goltsman et al. (2007), Alonso and Matouschek (2008) and Kovac and Mylovanov (2009), among others.

[5]Huber and Shipan (2006) survey the political science literature on this topic. Since the seminal work of Crawford and Sobel (1982), the game theoretic literature on communication in settings with private information and conflicting interests has flourished by pursuing the analysis of this "Ally Principle". Those analysis focused on the cheap talk timing where informed agents move first and sometimes imposed further exogenous constraints to compare alternative game forms and institutions (see Gilligan and Krehbiel, 1987 and Dessein, 2002 for instance).

robustness check of a by now large body of works but also because of its relevance for real world problems. Indeed, most economic examples presented above involve control of an agent in a multi-dimensional context. To give a first pass on these questions, the present paper investigates the form of optimal communication mechanisms when the agent's activity is two-dimensional.

Although the principal has still no transfers to better align his objectives with those of the agent, he may now trade off distortions on different aspects of the agent's overall activity to facilitate information revelation. To do so, the principal could for instance recoup the cost of implementing extra distortions on one dimension, favoring thereby the agent, through the benefits of lesser distortions on the other dimension, moving de facto this latter decision towards his own ideal point. We characterize the optimal communication mechanism in such an environment, where the principal's biases with the agent's ideal point on each dimension of his activities differ. Compared with the one-dimensional case, the design of the optimal communication mechanism brings new important insights. This optimal mechanism never exhibits any pooling and the trade-off between rules and discretion no longer exists. Instead, the optimal mechanism tights the agent's choices on each dimension through a smooth delegation set that never crosses the agent's ideal points.

The design of the optimal communication mechanism is complex because the absence of monetary transfers introduces a strong non-linearity in the principal's optimization problem. Intuitively, what matters from the incentive viewpoint is, on the one hand, the average decision that the principal would like to implement and, on the other hand, its "variance", i.e., how far apart the levels of each activity are one from the other. That "variance" plays a role similar albeit different than what transfers play in standard models with quasi-linear preferences and monetary payments. The similarity comes from the fact that, as transfers in usual screening models, more "variance" facilitates screening and information revelation. The difference is that increasing that "variance" is costly not only for the agent but also for the principal himself who has also a quadratic loss function. This is not so in standard models with quasi-linear preferences and monetary payments where there is a conflict between the principal and the agent on where the money should be: one dollar left to the agent costs to the principal and vice versa. One of our main findings is that the principal is always ready to sacrifice a bit on that side to facilitate screening and better align the agent's average decision with his own.

From a technical viewpoint, this non-linearity due to the absence of monetary payments renders the characterization of the optimal communication mechanism rather complex.[6] We use results from the calculus of variations (Clarke, 1990) to ensure existence of a solution and derive sufficient and necessary conditions for optimality.

[6]The absence of monetary transfers makes the contracting problem looks a bit like those that are solved in the optimal taxation literature (Mirrlees (1971)). The techniques of this literature cannot nevertheless be directly imported into our framework. Even with quadratic payoffs and once one has eliminated the "variance" of the decision from the principal and agent's objectives using incentive constraints, the principal's objective function which depends on the agent's payoff and its type derivative may not be everywhere Lipsichtz-continuous contrary to what happens in the optimal taxation literature.

Related Literature. Melumad and Shibano (1991) provided a significant analysis of the delegation problem with quadratic payoffs and a uniform types distribution in contexts where no transfers are available and where the uninformed party (the principal) commits to a communication mechanism with the uninformed party (the agent). Martimort and Semenov (2006) and Alonso and Matouschek (2008) characterized settings where simple connected delegation sets are optimal, a feature that was a priori assumed in Holmström (1984), Armstrong (1994) and Baron (2000) for instance. In particular, Alonso and Matouschek (2008) have proposed powerful optimization techniques which do not rely on calculus of variations but are highly specific to the one-dimensional setting. Our multi-dimensional model restores a role for a careful use of the calculus of variations in mechanism design problems without transfers. Alonso and Matouschek (2007) have brought the standard delegation model to a dynamic context where the principal and the agent repeatedly interact. Focusing on dominant strategy to get a sharp characterization of the set of incentive feasible allocations, Martimort and Semenov (2008) have instead extended this mechanism design approach to the case of multiple privately informed agents (lobbyists) dealing with a single principal (a Legislature) in a political economy context where the principal chooses a one-dimensional policy.[7] None of these papers has addressed the design of multi-dimensional communication mechanisms with a single agent when the principal has the ability to commit to a mechanism.

Organization of the paper. Section 2 presents the model and the by-now standard result where a one-dimensional activity of the agent is controlled by the principal. Section 3 presents some preliminary results on incentive compatibility and assesses the performances of simple and intuitive mechanisms that take into account the new possibilities that incentive compatibility in multi-dimensional environments open. Section 4 is the core of the paper. We formulate the design problem using advanced tools of the calculus of variations and we derive the optimal multi-dimensional mechanism. Some robustness checks are provided in Section 5. Proofs are relegated to the Appendix.

2 The Model

A principal controls two actions, x_1 and x_2, undertaken by a single agent on his behalf. We denote by (x_1, x_2) the bi-dimensional vector of those actions. For simplicity, those actions lie in a compact set $\mathcal{K} = [-K, K] \subseteq \mathbb{R}$ for K large enough.

Preferences. Utility functions are single-peaked, quadratic and respectively given for the principal and his agent by:[8]

$$V(x_1, x_2, \theta) = -\frac{1}{2}\sum_{i=1}^{2}(x_i - \theta - \delta_i)^2 \quad \text{and} \quad U(x_1, x_2, \theta) = -\frac{1}{2}\sum_{i=1}^{2}(x_i - \theta)^2.$$

[7]Austen-Smith (1993), Battaglini (2002, 2004), Krishna and Morgan (2001), Levy and Razin (2007), and Ambrus and Takahashi (2008) have instead considered cheap talk settings with multiple privately informed senders.

[8]The choice of quadratic utility functions is standard in the literature. This assumption is reasonable if one sees it as a Taylor approximation of more general utility functions in a context where actions would not move much around the agent's ideal point.

With those preferences, the agent's ideal point on each dimension is θ whereas the principal has an ideal point located at $\theta + \delta_i$ in dimension i, $i = 1, 2$. The principal is biased in the same direction on both dimensions but his preferences may be more or less congruent with those of the agent, i.e., $0 \leq \delta_1 \leq \delta_2$. To avoid trivial situations where the principal offers pooling allocations whatever the agent's type, we assume $\delta_2 < \frac{1}{2}$. For further references, we denote by $\Delta \equiv \delta_2 - \delta_1$ the difference in biases between the two dimensions and by $\delta \equiv \frac{\delta_1+\delta_2}{2}$ the average bias.

Information. The agent has private information on his ideal point θ (or type), that is drawn from a uniform distribution on $\Theta = [0, 1]$.[9] The principal is not informed about the agent's type.

Examples. Sometimes up to some renormalizations, several economic examples fit that bare bone description:

Regulating a multiproduct monopolist. Consider a multiproduct monopolist that produces two goods in quantities q_1 and q_2 at constant marginal cost β which is his private information. There are no scope or scale economies. The inverse demand for each good has a constant elasticity $P_i(q_i) = A_i q_i^{-\lambda_i}$ for some non-negative A_i and with $\lambda_i \in (0, 1)$. In this context, efficient and monopoly productions are respectively given by $q_i^* = \left(\frac{\beta}{A_i}\right)^{-1/\lambda_i}$ and $q_i^M = \left(\frac{\beta}{A_i(1-\lambda_i)}\right)^{-1/\lambda_i}$. Note that $\lambda_i \ln q_i^* = \lambda_i \ln q_i^M - \ln(1 - \lambda_i)$. Written in terms of $x_i = \lambda_i \ln q_i - \ln A_i - \ln(1 - \lambda_i)$ and $\theta = -\ln\beta$, we could think of this outcome as being implemented when the regulator's preferences are $-\frac{1}{2}\sum_{i=1}^{2}(x_i - \theta - \delta_i)^2$, where $\delta_i = -\ln(1 - \lambda_i)$, whereas the manager running the monopoly wants instead to maximize $-\frac{1}{2}\sum_{i=1}^{2}(x_i - \theta)^2$ in the absence of regulation. ■

Macroeconomic stabilization. Consider a government or a public body like a Central Bank (the agent) interested in stabilizing output, budget and inflation around some specified targets. Typically that government wants to minimize the loss function $(y - \bar{y}_G)^2 + (g - \bar{g}_G)^2 + \pi^2$ where y, g and π are respectively the logs of output, public expenditures and inflation. It does so by means of using a capital tax τ and the rate of money growth $m = \pi$. Public expenditures are $g = \pi + \tau$ whereas output, taxes and inflation are linked through a Philips curve relationship: $y = \pi - \tau - \pi_e - \beta$ where π_e is the expected inflation rate and β a macroeconomic shock privately known by the government. Overall, the government's loss function can be rewritten as (up to a constant)

$$3\pi^2 - 2\pi(\beta + \bar{g}_G + \pi_e + \bar{y}_G) + 2\tau^2 + 2\tau(\beta - \bar{g}_G + \pi_e + \bar{y}_G).$$

Note that this objective is such that there is no interaction between τ and π. Letting $\theta = \beta + \bar{g}_G + \pi_e + \bar{y}_G$, $x_1 = -2\tau + 2\bar{g}_G$ and $x_2 = 3\pi$, the government's objective can be written as minimizing $\frac{1}{2}(x_1 - \theta)^2 + \frac{1}{3}(x_2 - \theta)^2$. Society (the principal) as a whole has a similar loss function but with different parameters $\bar{y}_S$ and $\bar{g}_S$ and thus different ideal points so that the loss function can be written as $\frac{1}{2}(x_1 - \theta - \delta_1)^2 + \frac{1}{3}(x_2 - \theta - \delta_2)^2$ where $\delta_1 = \bar{y}_S - \bar{g}_S - \bar{y}_G + \bar{g}_G$ and $\delta_2 = \bar{y}_S + \bar{g}_S - \bar{y}_G - \bar{g}_G$. Notice that, in this example, quadratic losses do not have the same weights on the two dimensions. It is straightforward to extend our model to this setting as long as those weights are common to

[9] The characterization of the contractual outcomes would be untractable if we were assuming other distributions.

the principal and the agent. ■

Mechanisms. The principal controls the whole vector of the agent's activities (x_1, x_2). From the Revelation Principle (Myerson, 1982), there is no loss of generality in restricting the analysis to direct communication mechanisms stipulating (maybe stochastic[10]) decisions as functions of the agent's report on his type. Any deterministic communication mechanism is a mapping $x(\cdot) = \{x_1(\cdot), x_2(\cdot)\} : \Theta \to \mathcal{K}^2$.

Timing. The communication game between the principal and his agent unfolds as follows:

- ⋆ First, the agent learns the state of nature θ.
- ⋆ Second, the principal commits to a mechanism $x(\cdot)$.
- ⋆ Third, the agent communicates with the principal by sending a message $\hat{\theta} \in \Theta$ and the requested actions $(x_1(\hat{\theta}), x_2(\hat{\theta}))$ are implemented.

Benchmark: the one-dimensional activity case. For further references, let us consider the case where the principal controls a single decision x_i, for some $i \in \{1, 2\}$.

Proposition 1 (One-Dimensional Activity) *In the one-dimensional case, the optimal communication mechanism $x_i^O(\theta)$ is given by:*

$$x_i^O(\theta) = \max\left\{\theta, \theta_i^O\right\}, \ \textit{where } \theta_i^O = 2\delta_i < 1. \tag{1}$$

In the one-dimensional case, the optimal communication mechanism has a simple structure: The optimal action corresponds to the agent's ideal point if it is large enough and is otherwise independent of the agent's type. The intuition for why the optimal mechanism does not need to entail a cap is straightforward. Since he prefers lower levels of activity than the principal, the agent wants to lie "downward" and report that the state of nature θ is less than what it really is so that the principal would recommend a lower action. Reporting high values of θ is unlikely to reflect lying behavior and the principal finds it useless to impose a cap on activity levels.

This outcome can be easily achieved by means of a simple *delegation set*. Instead of using a direct revelation mechanism and communicating with the agent, the principal could as well offer a menu of options $\mathcal{D}_i = \left[\theta_i^O, +\infty\right)$ and let the agent freely choose within this set. When the floor θ_i^O is not binding, the agent is not constrained by the principal's choice of a delegation set and everything happens as if he had full discretion in choosing his own ideal point. When the floor is instead binding, the agent is constrained and cannot choose his bliss point which is too low compared with what the principal would implement himself.

The optimal communication mechanism trades off the benefits of flexibility (the agent choosing sometimes a state-dependent action) against the loss of control it implies (this state-dependent

[10] We postpone to Subsection 5.1 the analysis of stochastic mechanisms proving there their suboptimality.

action being different from the principal's ideal point). Setting a floor θ_i^O limits the agent's discretion and reduces the loss of control. Clearly, θ_i^O increases with δ_i meaning that a less rigid rule is chosen when the conflict of interests between the principal and the agent is less pronounced: the so-called *"Ally Principle"*. We will see later how this trade-off between rules and discretion and the associated comparative statics are modified in a multi-dimensional setting.

3 Preliminary Results

In the multi-dimensional case, incentive compatibility constraints can be written as:

$$\theta \in \arg\max_{\hat{\theta}\in\Theta} -\frac{1}{2}\sum_{i=1}^{2}\left(x_i(\hat{\theta})-\theta\right)^2.$$

Lemma 1 *The necessary and sufficient condition for incentive compatibility is that $\sum_{i=1}^{2} x_i(\theta)$ is non-decreasing in θ and thus a.e. differentiable in θ. At any differentiability point, we have:*

$$\sum_{i=1}^{2} \dot{x}_i(\theta) \geq 0, \tag{2}$$

$$\sum_{i=1}^{2} \dot{x}_i(\theta)(x_i(\theta)-\theta) = 0. \tag{3}$$

In this multi-dimensional world, the principal can now use both $x_1(\cdot)$ and $x_2(\cdot)$ to screen the agent's preferences. To understand how it can be so, it is useful first to observe that the principal could at least offer the optimal one-dimensional communication mechanisms he would offer for each dimension, namely the pair of mechanisms described in Proposition 1. Although communication mechanisms that would satisfy (1) for $i = 1, 2$ also satisfy (2) and (3), more communication mechanisms are now feasible. By trading off distortions along each dimension or by choosing actions that vary in opposite directions on each dimension as the agent's type changes but still are both biased towards his own ideal points, the principal can introduce countervailing incentives which might facilitate information revelation.[11]

This characterization of incentive compatible allocations already gives some powerful insights on the properties of mechanisms by looking at a couple of simple communication mechanisms.

Example 1 Consider the linear communication mechanism $\{x_1^\alpha(\theta), x_2^\alpha(\theta)\}_{\theta\in\Theta}$ such that $x_1^\alpha(\theta) = \theta-\alpha$ and $x_2^\alpha(\theta) = \theta+\alpha$ where α is a fixed number. This mechanism is incentive compatible since it satisfies both (2) and (3). The best of such communication mechanisms maximizes the principal's

[11] The literature on countervailing incentives (Lewis and Sappington, 1989a,b and Laffont and Martimort, 2002, Chapter 3 among others) has been developed in settings with monetary transfers. Sometimes those models generate pooling as an optimal response to incentives to over- and under-report types as in Lewis and Sappington (1989b). On the contrary, in our model pooling is never an issue as we show below.

profit, i.e., α should be optimally chosen so that any concession made by the principal on x_1 by moving this decision closer to the agent's own ideal point is compensated by an equal shift in x_2 in the direction of the principal's ideal point. Typically, a uniform distortion $\alpha = \frac{\Delta}{2}$ does the trick since

$$\arg\min_{\alpha} \int_0^1 \left(\sum_{i=1}^{2} (x_i^{\alpha}(\theta) - \theta - \delta_i)^2 \right) d\theta = \arg\min_{\alpha} (\alpha + \delta_1)^2 + (\alpha - \delta_2)^2 = \frac{\Delta}{2}.$$

■

Still the principal can find that such decision rules is too close to the agent's ideal points. The mechanism $\{x_1^{\alpha}(\theta), x_2^{\alpha}(\theta)\}_{\theta\in\Theta}$ can be improved upon by introducing a pooling area as in the one-dimensional case. This trick is used in the next example to improve the principal's expected payoff.

Example 2 Consider the incentive compatible mechanism $\{\tilde{x}_1(\theta), \tilde{x}_2(\theta)\}_{\theta\in\Theta}$ defined as:

$$\tilde{x}_1(\theta) = \begin{cases} \tilde{\theta} - \frac{\Delta}{2} & \text{if } \theta \leq \tilde{\theta} \\ \theta - \frac{\Delta}{2} & \text{otherwise} \end{cases} \text{ and } \tilde{x}_2(\theta) = \begin{cases} \tilde{\theta} + \frac{\Delta}{2} & \text{if } \theta \leq \tilde{\theta} \\ \theta + \frac{\Delta}{2} & \text{otherwise.} \end{cases} \tag{4}$$

This new mechanism is obtained by piecing together a floor on policy for $\theta \leq \tilde{\theta}$ and a mechanism trading off distortions on each dimension for $\theta \geq \tilde{\theta}$. The optimal mechanism within this class, that we denote thereafter $\{\tilde{x}_1^*(\theta), \tilde{x}_2^*(\theta)\}_{\theta\in\Theta}$, is such that

$$\tilde{\theta}^* = \arg\min_{\tilde{\theta}\in\Theta} \int_0^1 \left(\sum_{i=1}^{2} (\tilde{x}_i(\theta) - \theta - \delta_i)^2 \right) d\theta = \arg\min_{\tilde{\theta}\in\Theta} \int_0^{\tilde{\theta}} (\tilde{\theta} - \theta - \delta)^2 d\theta + \int_{\tilde{\theta}}^1 \delta^2 d\theta = 2\delta.$$

This mechanism has a non-trivial pooling area since $2\delta < 1$. The principal limits now the pooling area to an average between the pooling areas that he would choose when designing an optimal mechanism on each dimension separately. This shows how distortions on each dimension are traded one against the other. ■

The next proposition shows that the principal's loss is greater with the "naive" communication mechanism $\{x_1^O(\theta), x_2^O(\theta)\}_{\theta\in\Theta}$ given by Proposition 1, that treats each dimension separately, than with the simple communication mechanism $\{\tilde{x}_1^*(\theta), \tilde{x}_2^*(\theta)\}_{\theta\in\Theta}$ introduced in Example 2 above as long as the average bias is not too high ($\delta \leq 1/4$). The intuition is that, when δ is small, the range of separating allocations is large, so the benefits of distortions on x_1 and x_2 is large. On the contrary, when δ is greater, the naive mechanism is more efficient because it allows to set two different pooling areas $[0, 2\delta_1]$ for $x_1^O(\cdot)$ and $[0, 2\delta_2]$ for $x_2^O(\cdot)$, while the pooling area $[0, 2\delta]$ for $\{\tilde{x}_1^*(\cdot), \tilde{x}_2^*(\cdot)\}$ is necessarily common on both dimensions. This will become more apparent latter when we will show that the optimal mechanism is closer to $\{\tilde{x}_1^*(\theta), \tilde{x}_2^*(\theta)\}_{\theta\in\Theta}$ on the upper tail of the distribution, but closer to $\{x_1^O(\theta), x_2^O(\theta)\}_{\theta\in\Theta}$ on the lower tail of the distribution.

Proposition 2 *The principal's loss is greater with the naive mechanism* $\{x_1^O(\theta), x_2^O(\theta)\}_{\theta\in\Theta}$ *than with the simple mechanism* $\{\tilde{x}_1^*(\theta), \tilde{x}_2^*(\theta)\}_{\theta\in\Theta}$ *if and only if* $\delta \leq 1/4$.

Example 2 is instructive because it stresses two aspects of optimal mechanisms that our more general analysis will confirm. First, the principal trades off distortions on some interval of types. Second, decision rules should be rather flat on the lower tail of the distribution. However, contrary to the simple mechanism $\{\tilde{x}_1^*(\theta), \tilde{x}_2^*(\theta)\}_{\theta\in\Theta}$, the optimal mechanism will not exhibit any pooling and one action will be decreasing on the upper and lower tails of the distribution.

4 Optimal Multi-Dimensional Mechanism

4.1 Changing Variables

To characterize the optimal mechanism, it is useful to use a new set of variables to reparameterize our problem. This transformation shall not only bring new insights on the nature of the economic problem but it will also allow us to easily show later on that stochastic or discontinuous mechanisms are not optimal. Consider thus the following two extra auxiliary variables which are the average decision and a measure of the "variance" of those decisions:

$$x(\theta) \equiv \frac{1}{2}\sum_{i=1}^{2} x_i(\theta) \text{ and } t(\theta) \equiv \frac{1}{2}\sum_{i=1}^{2}(x_i(\theta) - x(\theta))^2 = \frac{1}{4}(x_2(\theta) - x_1(\theta))^2. \tag{5}$$

Solving this system of equations for $x_1(\theta)$ and $x_2(\theta)$ yields immediately:

$$x_1(\theta) = x(\theta) - \sqrt{t(\theta)} \quad \text{and} \quad x_2(\theta) = x(\theta) + \sqrt{t(\theta)}. \tag{6}$$

Define now the agent's non-positive *information rent* $U(\theta)$ as:

$$U(\theta) \equiv \max_{\hat{\theta}\in\Theta} -\frac{1}{2}\left(\sum_{i=1}^{2}\left(x_i(\hat{\theta}) - \theta\right)^2\right).$$

Using (6) and incentive compatibility, we rewrite:

$$U(\theta) = -(x(\theta) - \theta)^2 - t(\theta) = \max_{\hat{\theta}\in\Theta} -(x(\hat{\theta}) - \theta)^2 - t(\hat{\theta}). \tag{7}$$

With this formulation, the agent's rent depends now only on the screening variables through the average decision $x(\theta)$ and their variance $t(\theta)$. This utility function becomes "quasi-linear" with the "transfer" $t(\theta)$ measuring the cost for the agent of choosing different decisions along each dimension. The technical difficulty that we will face in the sequel comes from the fact that this transfer does not enter linearly into the principal's objective.

The average decision $x(\theta)$ has an impact on the agent's marginal utility which depends on his realized type. It can thus be used as a screening variable as in standard screening models. Clearly, an agent with type θ may be tempted to lie downward to move the average decision closer to his own ideal point. The principal can make that strategy less attractive by putting more "risk" on the agent, increasing the spread between decisions for the lowest types.[12]

As usual in screening problems with quasi-linear utility functions, the incentive compatibility conditions (2) and (3) can be restated in terms of the properties of the pair $(U(\theta), x(\theta))$.

Lemma 2 *The information rent $U(\theta)$ is absolutely continuous with a first derivative defined almost everywhere and, at any differentiability point:*

$$\dot{U}(\theta) = 2(x(\theta) - \theta). \tag{8}$$

The average decision $x(\theta)$ is non-decreasing and thus almost everywhere differentiable with, at any differentiability point:

$$\dot{x}(\theta) \geq 0. \tag{9}$$

Note that $t(\theta) \geq 0$ implies

$$-U(\theta) - \frac{\left(\dot{U}(\theta)\right)^2}{4} \geq 0, \tag{10}$$

with an equality only when $x_1(\theta) = x_2(\theta) = x(\theta)$, i.e., when both decisions are equal.[13]

4.2 Design and Properties

With the new set of variables, we rewrite the principal's loss in each state of nature θ as:

$$\frac{1}{2}\sum_{i=1}^{2}(x_i(\theta) - \theta - \delta_i)^2 \equiv L_\Delta(U(\theta), \dot{U}(\theta)) = -U(\theta) - \delta\dot{U}(\theta) - \Delta\sqrt{-U(\theta) - \frac{(\dot{U}(\theta))^2}{4} + \delta^2 + \frac{\Delta^2}{4}}.$$

This yields the following expression of the principal's relaxed problem neglecting for the time being the monotonicity condition on $x(\theta)$ that will be checked ex post:

$$(\mathcal{P}_\Delta): \quad \min_{U \in W^{1,1}(\Theta)} \int_0^1 L_\Delta(U(\theta), \dot{U}(\theta))d\theta,$$

where $W^{1,1}(\Theta)$ denotes the set of absolutely continuous arcs on Θ. In the parlance of the calculus of variations, $(\mathcal{P}_\Delta)$ is actually a Bolza problem with free end-points (see Clarke, 1990, Chapter 4). It is non-standard because the functional $L_\Delta(s, v)$ even though it is continuous and strictly convex in (s, v) is not everywhere differentiable (or even Lipschitz), especially at points where $-s - \frac{v^2}{4} = 0$ if any such point exists on an admissible curve where $v(\theta) = \dot{U}(\theta)$ and $s(\theta) = U(\theta)$.

[12]The principal-agent literature has already stressed that a principal can use the agent's risk-aversion to ease incentives (see, e.g., Arnott and Stiglitz, 1988) by for instance using stochastic mechanisms. Introducing some variance in the agent's decisions in a model with quadratic payoffs has a similar flavor. Subsection 5.1 shows that stochastic mechanisms are suboptimal in our framework.

[13]This is typically the case in the one-dimensional world. The fact that (10) is not an equality reflects the multi-dimensionality aspect of the screening problem.

We now proceed as follows. First, we prove existence of an optimal arc in $W^{1,1}(\Theta)$. Second, we characterize this arc by means of a second-order Euler-Lagrange equation. Third, a first quadrature tells us that such solution solves a first-order differential equation known up to a constant. Finally, we impose conditions on that constant so that the monotonicity condition (9) always holds.

Lemma 3 *A solution $U^*(\cdot)$ to $(\mathcal{P}_\Delta)$ exists.*

Once such solution is known, the pair of decision rules $\{x_1^*(\cdot), x_2^*(\cdot)\}$ is recovered using the formulae:

$$x_1^*(\theta) = \theta + \frac{\dot{U}^*(\theta)}{2} - \sqrt{-U^*(\theta) - \frac{(\dot{U}^*(\theta))^2}{4}} \text{ and } x_2^*(\theta) = \theta + \frac{\dot{U}^*(\theta)}{2} + \sqrt{-U^*(\theta) - \frac{(\dot{U}^*(\theta))^2}{4}}. \quad (11)$$

Proposition 3 *An optimal arc $U^*(\cdot)$ is such that:*

- *The following Euler-Lagrange equation holds at any interior point of differentiability:*

$$\frac{\partial L_\Delta}{\partial U}(U^*(\theta), \dot{U}^*(\theta)) = \frac{d}{d\theta}\left(\frac{\partial L_\Delta}{\partial \dot{U}}(U^*(\theta), \dot{U}^*(\theta))\right); \quad (12)$$

- *The following free end-points hold on the boundaries of the interval $[0,1]$:*

$$\frac{\partial L_\Delta}{\partial \dot{U}}(U^*(\theta), \dot{U}^*(\theta))|_{\theta=0} = \frac{\partial L_\Delta}{\partial \dot{U}}(U^*(\theta), \dot{U}^*(\theta))|_{\theta=1} = 0; \quad (13)$$

- *$U^*(\theta)$ is continuously differentiable, and thus $x_1^*(\theta)$ and $x_2^*(\theta)$ are continuous.*

The next proposition investigates the nature of the solution to the second-order ordinary differential equation (12) by obtaining a first quadrature parameterized by some integration constant $\lambda \in \mathbb{R}$. This constant must be non-positive to ensure that the second-order condition (9) holds.

Proposition 4 *For each solution $U(\theta, \lambda)$ to (12) which is everywhere negative and satisfies (9), there exists $\lambda \in \mathbb{R}_-$ such that*[14]

$$\dot{U}(\theta, \lambda) = 2\sqrt{-U(\theta, \lambda) - \Delta^2\left(\frac{U(\theta, \lambda)}{U(\theta, \lambda) + \lambda}\right)^2}, \quad (14)$$

and

$$(U(\theta, \lambda) + \lambda)^2 + \Delta^2 U(\theta, \lambda) > 0 \textit{ for all } \theta \in \Theta. \quad (15)$$

[14] It is important to note that the differential equation (14) may a priori have a singularity and more than one solution going through a given point. This might be the case when, for such solution, there exists θ_0 such that $U(\theta_0, \lambda) + \Delta^2\left(\frac{U(\theta_0,\lambda)}{U(\theta_0,\lambda)+\lambda}\right)^2 = 0$. Indeed, the right-hand side of (14) fails to be Lipschitz at such a point. It turns out that this possibility does not arise for the optimal mechanism described below because a careful choice of λ ensures that the condition (15) holds everywhere on the optimal path.

We are now ready to characterize the optimal mechanism in the multi-dimensional case.

Theorem 1 (Two-Dimensional Activity.) *Assume the principal controls the two decisions x_1 and x_2 of the agent. When $\Delta > 0$ the optimal communication mechanism is such that:*

- *Optimal decisions on each dimension are never equal to the agent's ideal points:*
$$x_1^*(\theta) = x^*(\theta) - \frac{\Delta U^*(\theta)}{U^*(\theta) + \lambda^*} < x_2^*(\theta) = x^*(\theta) + \frac{\Delta U^*(\theta)}{U^*(\theta) + \lambda^*}, \tag{16}$$
with $\lambda^* \in (-\frac{\Delta^2}{4} - \delta^2, -\frac{\Delta^2}{4})$ and $x^*(\theta) = \theta + \frac{\dot{U}^*(\theta)}{2}$;
- *The rent profile $U^*(\theta)$ is everywhere negative, strictly increasing and solves (14) for λ^*;*
- *There is no pooling area. Monotonicity conditions are satisfied everywhere $\dot{x}^*(\theta) > 0$.*

When $\Delta = 0$, the optimal mechanism coincides with that described in Proposition 1 for the one-dimensional problem, with

$$U_0^*(\theta) = -(\min\{\theta - 2\delta, 0\})^2, \quad x_0^*(\theta) = \max\{\theta, 2\delta\} \text{ and } \lambda^* = 0. \tag{17}$$

The intuition for the optimality of the naive mechanism when $\delta_1 = \delta_2$ is that, exactly as in Examples 1 and 2, there is no gain for the principal to trade off distortions on each dimension. Beyond this special case, several features of the optimal mechanism are worth being stressed when $\Delta > 0$. First, in sharp contrast with the one-dimensional case, the agent's ideal points are never chosen at the optimal mechanism. By trading off distortions on each dimension, the principal is always able to induce truthtelling without making the agent residual claimant for those decisions. Second, even when the agent's ideal point is on the lower tail of the distribution, there is no need to offer a pooling contract; $x^*(\theta)$ is monotonically increasing everywhere. Again, there is always a better option than a pooling contract which is to trade off distortions on each decision even if it is marginally so.

Corollary 1 *For any $\Delta > 0$, we have $[2\delta, 1] \subsetneq [x^*(0), x^*(1)]$. More precisely, there exists $\theta^*(\Delta) \in (0, 2\delta)$ such that:*

$$x^*(\theta) < 2\delta \text{ if and only if } \theta \leq \theta^*(\Delta).$$

Moreover, we have also:

$$x^*(\theta) > \theta \text{ for all } \theta \in \Theta.$$

First, the average decision is systematically spread over a greater interval than if the principal was restricted to offer the simple mechanism $\{\tilde{x}_1^*(\theta), \tilde{x}_2^*(\theta)\}_{\theta \in \Theta}$ or the naive mechanism $\{x_1^O(\theta), x_2^O(\theta)\}_{\theta \in \Theta}$. There would be a systematic bias introduced by restricting the analysis to those simple mechanisms. Second, Corollary 1 also shows that the benefits of fine-tuning the distortions on decisions allows the principal to move up the average decision further away from the agent's ideal points. These features are illustrated in Figure 1 which compares the average decision $\tilde{x}^*(\theta) = \frac{1}{2}(\tilde{x}_1^*(\theta) + \tilde{x}_2^*(\theta)) = \max\{2\delta, \theta\}$ with the optimal average decision $x^*(\theta)$ for a fixed average bias δ and different values of Δ. The agent's information rent under the optimal mechanism is represented in Figure 2.

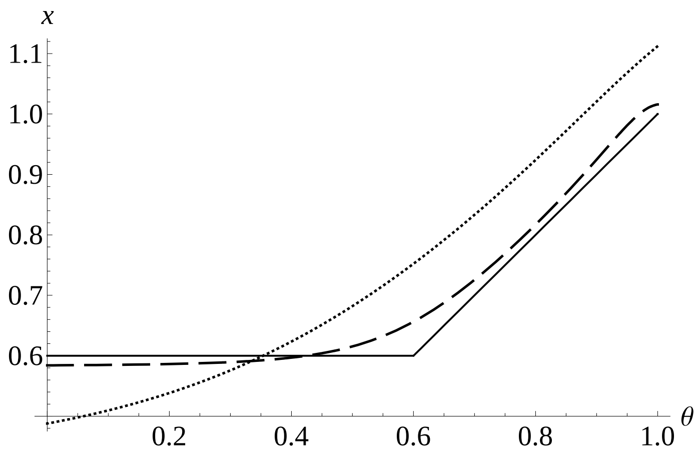

Figure 1: Average decision $x^*(\theta)$ when $\delta = 0.3$, and $\Delta = 0.6$ (dotted line), $\Delta = 0.2$ (dashed line) and $\Delta = 0$ (plain line, which coincides with $\tilde{x}^*(\theta)$).

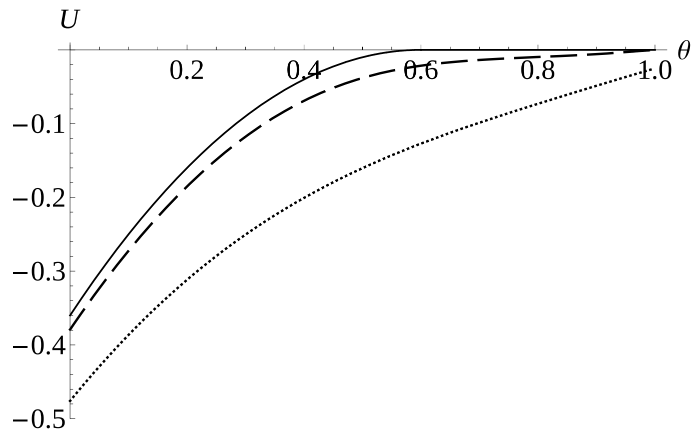

Figure 2: Agent's information rent $U^*(\theta)$ when $\delta = 0.3$, and $\Delta = 0.6$ (dotted line), $\Delta = 0.2$ (dashed line), and $\Delta = 0$ (plain line) .

The distortions on each dimension are more complex, as illustrated by Figure 3. While $x_2^*(\theta)$ is always strictly greater than θ, it is not always increasing, while $x_1^*(\theta)$ is strictly increasing over $[0, 1]$ but not always greater than θ. In addition, for the incentive compatibility constraint (3) to be satisfied, $x_2^*(\theta)$ should be strictly decreasing if and only if $x_1^*(\theta)$ is larger than θ. This feature of the optimal mechanism is general, and is summarized in the next corollary.

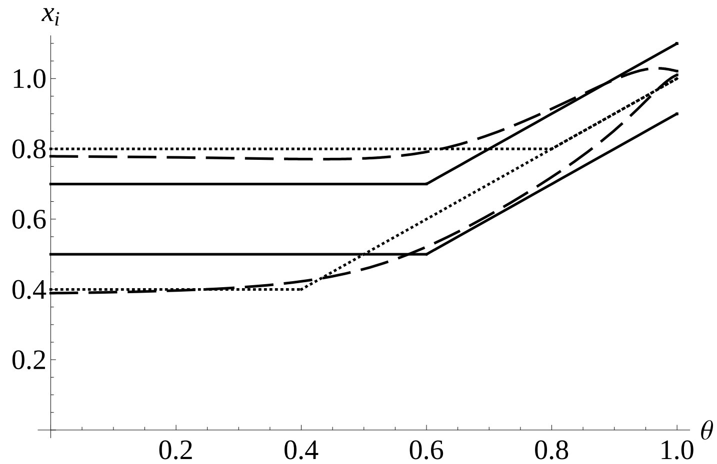

Figure 3: One-dimensional decisions $x_i^*(\theta)$ (dashed lines), $\tilde{x}_i^*(\theta)$ (plain lines) and $x_i^O(\theta)$ (dotted line), for $i = 1, 2$, when $\delta_1 = 0.2$ and $\delta_2 = 0.4$.

Corollary 2 *Assume that* $\Delta > 0$.

1. *For every* $\theta \in [0, 1]$, *we have* $\dot{x}_1^*(\theta) > 0$ *and* $x_2^*(\theta) > \theta$*;*

2. *For every* $\theta \in [0, 1]$, *we have* $x_1^*(\theta) > \theta$ *if and only if* $\dot{x}_2^*(\theta) < 0$*;*

3. $x_1^*(0) \geq 0$ *and* $\dot{x}_2^*(0) \leq 0$ *(with strict inequalities when* $\delta_1 > 0$*);*

4. $x_1^*(1) \geq 1$ *and* $\dot{x}_2^*(1) \leq 0$ *(with strict inequalities when* $\delta_1 > 0$*).*

The optimal mechanism is quite close to the naive mechanism on the lower tail of the distribution where there is little separation of types. Instead, for the upper tail of the distribution, this mechanism comes closer to the simple mechanism; then trading off distortions on each dimension becomes more attractive.

Even if simple delegation sets trading off inflexible rules and full discretion are no longer optimal, it is still true that a version of the Taxation Principle holds. The principal can implement the optimal communication mechanism by offering an indirect mechanism, i.e., a (continuous) curve in the (x_1, x_2) space constructed from the parametrization $\{x_1^*(\theta), x_2^*(\theta)\}_{\theta \in \Theta}$ and letting the agent free to pick any point on this curve. On Figure 4 we have represented in the (x_1, x_2) space such curve corresponding to the optimal mechanism. At the same time, this figure also features the indirect mechanisms corresponding to the simple mechanism $\{\tilde{x}_1^*(\theta), \tilde{x}_2^*(\theta)\}_{\theta \in \Theta}$ and the naive mechanism $\{x_1^O(\theta), x_2^O(\theta)\}_{\theta \in \Theta}$.

Since the design of the optimal mechanism looks rather complex, one may wonder whether simple mechanisms perform well and under which circumstances. The intuition is that, although the optimal mechanism requests full separation of types, it is only marginally so on the lower tail of the type distribution. In this respect, our next proposition shows that the simple mechanism

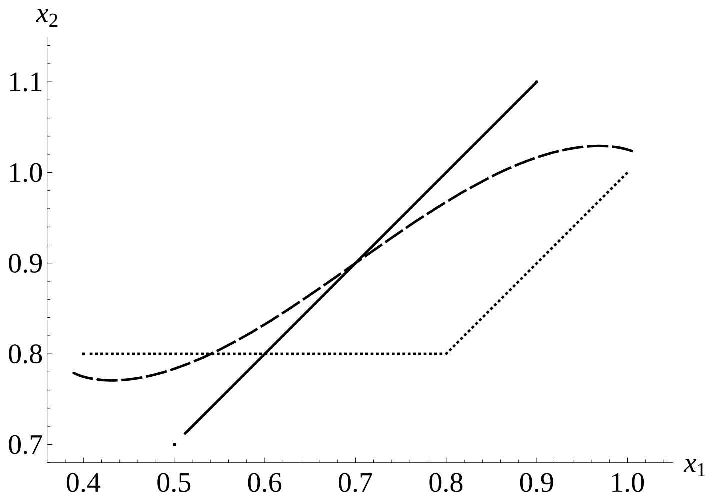

Figure 4: Delegation sets for the optimal mechanism (dashed lines), the simple mechanism $(\tilde{x}_1^*(\cdot), \tilde{x}_2^*(\cdot))$ (plain lines), and the naive mechanism $(x_1^O(\cdot), x_2^O(\cdot))$ (dotted lines) when $\delta_1 = 0.2$ and $\delta_2 = 0.4$.

$\{\tilde{x}_1^*(\theta), \tilde{x}_2^*(\theta)\}_{\theta\in\Theta}$ of Example 2 or the naive mechanism $\{x_1^O(\theta), x_2^O(\theta)\}_{\theta\in\Theta}$ that simply replicates the one-dimensional mechanisms perform quite well when Δ is small enough.

Proposition 5 *The principal's loss from using the simple mechanism $\{\tilde{x}_1^*(\theta), \tilde{x}_2^*(\theta)\}_{\theta\in\Theta}$ or the naive mechanism $\{x_1^O(\theta), x_2^O(\theta)\}_{\theta\in\Theta}$ instead of the optimal mechanism $\{x_1^*(\theta), x_2^*(\theta)\}_{\theta\in\Theta}$ is of order at most 2 in Δ:*

$$\int_0^1 L_\Delta(\tilde{U}^*(\theta), \dot{\tilde{U}}^*(\theta))d\theta - \int_0^1 L_\Delta(U^*(\theta), \dot{U}^*(\theta))d\theta \leq \frac{3\Delta^2}{4}, \tag{18}$$

$$\int_0^1 L_\Delta(U^O(\theta), \dot{U}^O(\theta))d\theta - \int_0^1 L_\Delta(U^*(\theta), \dot{U}^*(\theta))d\theta \leq (1-\delta)\Delta^2. \tag{19}$$

5 Extensions

This section develops some extensions of our basic framework and shows the robustness of some of our results.

5.1 Non-Optimality of Stochastic Mechanisms

Kovac and Mylovanov (2009) showed that the restriction to deterministic mechanisms is without loss of generality in the case of quadratic payoffs and a one-dimensional activity. This results clearly extends in our framework when $\Delta = 0$ *mutatis mutandis.* However, it holds also in our multi-dimensional context when $\Delta > 0$.

Proposition 6 *The optimal deterministic mechanism characterized in Theorem 1 cannot be improved by stochastic mechanisms.*

To provide incentives for truthtelling with stochastic mechanisms, the principal could a priori use random allocations and play on the variance of each decision, i.e., choose how decisions move around their expected values. Of course, he can still play with how decisions are spread as in our analysis of deterministic mechanisms. The second of those strategies has already been shown useful above. The first is suboptimal. The intuition is straightforward, the principal and the agent are both equally averse to such randomizations in allocations and there is no gain from using those that could not already have been achieved by playing on the spread between decisions alone.

5.2 Leaving No Discretion Is Generic

Our no-discretion result is highly robust. Indeed, this section shows that the principal never finds optimal to leave full discretion to the agent, leaving him free of choosing his ideal points, on a subset I with a no-empty interior whatever the everywhere positive and atomless density $f(\theta)$ on Θ.

Proposition 7 *Assume any everywhere positive and atomless density $f(\theta)$ on Θ. The optimal deterministic mechanism has never $x_1^*(\theta) = x_2^*(\theta) = \theta$ on any subset I with a no-empty interior.*

The intuition is straightforward. Suppose the contrary. The principal could, as in Example 1, move down x_1 and up x_2 by the same small amount on that interval still keeping the same average decision so that incentives for truthtelling are unchanged. Doing so yields a strict benefit to the principal who enjoys having decisions spread apart more than his agent.

6 Conclusion

Optimal multi-dimensional communication mechanisms are quite different from the simple delegation sets found in the one-dimensional case. The possibility of trading off distortions along each dimension of the agent's activities leads to fully separating allocations and makes simple delegation sets, which trade off inflexible rules and full discretion, suboptimal.

It would be worth investigating optimal mechanisms in more complex environments allowing more general utility functions and more general type distributions. Some relatively easy extensions should be to investigate optimal mechanisms when the principal and the agent value differently the losses on each dimension still keeping the quadratic structure. More generally, such extensions might meet strong technical difficulties coming from the impossibility to learn much from the Euler-Lagrange equation beyond the quadratic utility function/uniform type distribution investigated here. These difficulties will certainly have to be overcome by relying on unattractive numerical methods. Nevertheless, we feel quite confident about the robustness of our findings and especially on the fact that there should be more scope for fully separating allocations.

Second, it would be also important to extend our approach by allowing for multi-dimensional preferences: the agent's bliss points on each dimension of his activity being not necessarily perfectly correlated. This extension is also likely to meet strong technical difficulties but certainly deserves some attention to investigate the scope for having fully separating allocations in those contexts. For all those cases, we conjecture that the decomposition between the average decision and its variability will play a crucial role for contract design.

Appendix

Proof of Proposition 1. See Melumad and Shibano (1991). ∎

Proof of Lemma 1. *Necessity:* Incentive compatibility implies for all pairs $(\theta, \hat{\theta}) \in \Theta^2$:

$$\sum_{i=1}^{2}(x_i(\hat{\theta}) - \theta)^2 \geq \sum_{i=1}^{2}(x_i(\theta) - \theta)^2 \text{ and } \sum_{i=1}^{2}(x_i(\theta) - \hat{\theta})^2 \geq \sum_{i=1}^{2}(x_i(\hat{\theta}) - \hat{\theta})^2. \tag{A.1}$$

Summing those inequalities yields:

$$\sum_{i=1}^{2}(x_i(\theta) - x_i(\hat{\theta}))(\theta - \hat{\theta}) \geq 0. \tag{A.2}$$

Hence, $\sum_{i=1}^{2} x_i(\theta)$ is non-decreasing in θ. Therefore, it is almost everywhere differentiable with, at any differentiability point, a derivative such that (2) holds. At such a point, an incentive compatible mechanism must also satisfy the first-order condition of the agent's revelation problem, namely (3). Moreover, using (A.1), we get:

$$\sum_{i=1}^{2} x_i^2(\theta) - \sum_{i=1}^{2} x_i^2(\hat{\theta}) \geq 2\hat{\theta}\left(\sum_{i=1}^{2} x_i(\theta) - \sum_{i=1}^{2} x_i(\hat{\theta})\right).$$

Hence, $\sum_{i=1}^{2} x_i^2(\theta)$ is non-decreasing in θ when $\sum_{i=1}^{2} x_i(\theta)$ is itself non-decreasing. $\sum_{i=1}^{2} x_i^2(\theta)$ is thus almost everywhere differentiable.

Sufficiency: That $\sum_{i=1}^{2} x_i(\theta)$ is non-decreasing in θ is then also a sufficient condition for optimality.[15] Indeed, since $\sum_{i=1}^{2} x_i^2(\theta)$ and $\sum_{i=1}^{2} x_i(\theta)$ are both non-decreasing in θ and thus almost everywhere differentiable with, at any differentiability point, a derivative which is measurable, Theorem 3 in Royden (1988, p. 100) implies:

$$\sum_{i=1}^{2}(x_i(\hat{\theta}) - \theta)^2 - \sum_{i=1}^{2}(x_i(\theta) - \theta)^2 \geq \sum_{i=1}^{2} \int_{\theta}^{\hat{\theta}} \dot{x}_i(s)(x_i(s) - \theta)ds$$

$$= \sum_{i=1}^{2} \int_{\theta}^{\hat{\theta}} \dot{x}_i(s)(x_i(s) - s + s - \theta)ds = \sum_{i=1}^{2} \int_{\theta}^{\hat{\theta}} \dot{x}_i(s)(s - \theta)ds \geq 0,$$

where the last equality follows from (3) and the last inequality from (2). ∎

[15] Garcia (2005) provides an analysis of the multi-dimensional adverse selection model in a framework with quasi-linear utility functions but focuses a priori on differentiable mechanisms.

Proof of Proposition 2. We have

$$\int_0^1 L_\Delta(U^O(\theta),\dot U^O(\theta))d\theta-\int_0^1 L_\Delta(\tilde U^*(\theta),\dot{\tilde U}^*(\theta))d\theta = -\frac{2\delta_1^3}{3}-\frac{2\delta_2^3}{3}+\frac{4\delta^3}{3}+\frac{\Delta^2}{4} = (1/4-\delta)\Delta^2, \quad \text{(A.3)}$$

which is positive if and only if $\delta \leq 1/4$. ■

Proof of Lemma 2. The proof is standard and follows Milgrom and Segal (2002). ■

Proof of Lemma 3. We proceed along the lines of Clarke (1990, Chapter 4). Let us first define the extended-value Lagrangian

$$L_\Delta^*(s,v) = \begin{cases} L_\Delta(s,v) & \text{if } s \leq -\frac{v^2}{4}, \\ +\infty & \text{otherwise.} \end{cases}$$

As requested in Clarke (1990, p. 167), we observe that:

1. $L_\Delta^*(s,v)$ is $\mathcal{B}$-measurable where $\mathcal{B}$ denotes the $\sigma-$algebra of subsets of $\mathbb{R}\times\mathbb{R}$;
2. $L_\Delta^*(s,v)$ is lower-semi continuous;
3. $L_\Delta^*(s,v)$ is convex in v.

Define now the Hamiltonian as $H(s,p) = \sup_{v\in\mathbb{R}}\{pv - L_\Delta^*(s,v)\}$. When $s \leq -\frac{v^2}{4}$, $L_\Delta^*(s,v) = L_\Delta(s,v)$ is strictly convex in v and the maximum above is achieved for

$$p = \frac{\partial L_\Delta}{\partial v}(s,v). \quad \text{(A.4)}$$

This yields the maximand

$$v^* = 4(p+\delta)\sqrt{\frac{-s}{4(p+\delta)^2+\Delta^2}},$$

which gives

$$H(s,p) = \begin{cases} s+\sqrt{-s(4(p+\delta)^2+\Delta^2)}-\delta^2-\frac{\Delta^2}{4} & \text{if } s\leq 0, \\ -\infty & \text{otherwise.} \end{cases}$$

Note that $H(s,p)$ is differentiable on $\mathbb{R}_-\times\mathbb{R}$. We get the following inequality:

$$H(s,p) \leq |s| + \Delta\sqrt{|s|} + 2|p+\delta|\sqrt{|s|} - \delta^2 - \frac{\Delta^2}{4}.$$

Using now that $\sqrt{|s|} \leq 1+\frac{|s|}{2}$ and that $|p+\delta| \leq |p|+\delta$, we obtain finally:

$$H(s,p) \leq \Delta+2\delta-\delta^2-\frac{\Delta^2}{4}+2|p|+|s|\left(1+\delta+\frac{\Delta}{2}+|p|\right) \leq 2+2|p|+|s|\left(1+\delta+\frac{\Delta}{2}+|p|\right). \quad \text{(A.5)}$$

This is a "growth" condition on the Hamiltonian as requested in Clarke (1990, Theorem 4.1.3).

Lemma 4 *Clarke (1990). Assume that $L_\Delta^*(\cdot)$ satisfies conditions 1. to 3. above, that $H(\cdot)$ satisfies the "growth" equation (A.5) and that $\int_0^1 L_\Delta^*(U_0(\theta),\dot U_0(\theta))d\theta$ is finite for at least one admissible arc $U_0(\theta)$. Then, problem $\mathcal{P}_\Delta$ has a solution.*

It remains to show that $\int_0^1 L^*_\Delta(U_0(\theta), \dot{U}_0(\theta))d\theta$ is finite for at least one admissible arc $U_0(\theta)$. Take $U_0(\theta) = 0$ which corresponds to decisions $x_{10}(\theta) = x_{20}(\theta) = \theta$. This arc does the job and yields $\int_0^1 L^*_\Delta(U_0(\theta), \dot{U}_0(\theta))d\theta = \int_0^1 L^*_\Delta(U_0(\theta), \dot{U}_0(\theta))d\theta = \delta^2 + \frac{\Delta^2}{4}$. ■

Proof of Proposition 3. *Preliminaries:* We say that H satisfies the *strong Lipschitz condition* near an arc U if there exists $\epsilon > 0$ and a constant k such that for all $p \in \mathbb{R}$ and for all $(s_1, s_2) \in \mathbb{T}(U, \epsilon)$ the tube of radius ϵ centered on the arc U, the following inequality holds:

$$|H(s_1, p) - H(s_2, p)| \leq k(1 + |p|)|s_1 - s_2|. \tag{A.6}$$

This property holds in our context when there exists $\eta > 0$ such that $U(\theta) < -\eta$ for all θ (i.e., $U(\theta)$ is bounded away from zero which will be the case for the solution we exhibit below). Indeed, we have over the relevant range where $s_i \leq 0$:

$$|H(s_1, p) - H(s_2, p)| = |s_1 - s_2 + (\sqrt{-s_1} - \sqrt{-s_2})\sqrt{\Delta^2 + 4(p + \delta)^2}|.$$

Note that $|\sqrt{-s_1} - \sqrt{-s_2}| = \frac{|s_1 - s_2|}{2\sqrt{-s_0}}$ for some $s_0 \in \mathbb{T}(U, \epsilon)$ from the Mean-Value Theorem. Therefore, $|\sqrt{-s_1} - \sqrt{-s_2}| \leq \frac{|s_1 - s_2|}{2\sqrt{\eta - \epsilon}}$ for ϵ small enough. Hence, we get:

$$|H(s_1, p) - H(s_2, p)| \leq |s_1 - s_2| \left(1 + \frac{\sqrt{\Delta^2 + 4(p + \delta)^2}}{2\sqrt{\eta - \epsilon}}\right) \leq |s_1 - s_2| \left(1 + \frac{\Delta + 2(\delta + |p|)}{2\sqrt{\eta - \epsilon}}\right)$$

$$\leq \max\left\{1 + \frac{\Delta + 2\delta}{2\sqrt{\eta - \epsilon}}, \frac{1}{\sqrt{\eta - \epsilon}}\right\} |s_1 - s_2|(1 + |p|),$$

which is (A.6) with $k = \max\left\{1 + \frac{\Delta + 2\delta}{2\sqrt{\eta - \epsilon}}, \frac{1}{\sqrt{\eta - \epsilon}}\right\}$.

Euler equation and boundaries conditions: From Clarke (1990, Theorem 4.2.2, p.169), and since $L^*_\Delta(\cdot)$ satisfies conditions 1., 2., and 3. above and $H(\cdot)$ satisfies the strong Lipschitz condition (A.6), there exists an absolutely continuous arc $p(\cdot)$ such that the following conditions hold for the optimal arc $U^*(\theta)$.

- Optimality conditions for the Hamiltonian $H(\cdot)$:

$$-\dot{p}(\theta) = \frac{\partial H}{\partial s}(U^*(\theta), p(\theta)), \tag{A.7}$$

$$\dot{U}^*(\theta) = \frac{\partial H}{\partial p}(U^*(\theta), p(\theta)). \tag{A.8}$$

- Boundary conditions:

$$p(0) = p(1) = 0. \tag{A.9}$$

Using (A.4) yields $p(\theta) = \frac{\partial L_\Delta}{\partial \dot{U}}(U^*(\theta), \dot{U}^*(\theta))$. Differentiating with respect to θ, inserting into (A.7) and observing that $\frac{\partial H}{\partial s}(U^*(\theta), p(\theta)) = -\frac{\partial L_\Delta}{\partial U}(U^*(\theta), \dot{U}^*(\theta))$ yields (12). Finally, using again $p(\theta) = \frac{\partial L_\Delta}{\partial \dot{U}}(U^*(\theta), \dot{U}^*(\theta))$ yields (13).

Continuity: First observe that, a.e. on Θ, we have by definition

$$H(U^*(\theta), p(\theta)) = p(\theta)\dot{U}^*(\theta) - L_\Delta(U^*(\theta), \dot{U}^*(\theta)) \geq p(\theta)v - L_\Delta(U^*(\theta), v), \quad \forall\, v \leq 2\sqrt{-U^*(\theta)}. \tag{A.10}$$

If $\dot{U}$ is not continuous at some $\theta_0 \in (0,1)$, there exists an increasing sequence θ_n^- and a decreasing sequence θ_n^+ ($n \geq 1$) both converging towards θ_0, such that (A.10) applies at θ_n^-, θ_0 and θ_n^+, and (using monotonicity to get the strict inequality):

$$\lim_{n\to+\infty} \dot{U}^*(\theta_n^-) = \dot{U}^*(\theta_0^-) < \dot{U}^*(\theta_0^+) = \lim_{n\to+\infty} \dot{U}^*(\theta_n^+).$$

Because $L_\Delta(s,v)$ is continuous in (s,v) and $U^*(\theta)$ is absolutely continuous and thus continuous at θ_0, we have:

$$L_\Delta(U^*(\theta_0), v) = \lim_{n\to+\infty} L_\Delta(U^*(\theta_n^-), v) \text{ and } L_\Delta(U^*(\theta_0), \dot{U}^*(\theta_0^-)) = \lim_{n\to+\infty} L_\Delta(U^*(\theta_n^-), \dot{U}^*(\theta_n^-)). \tag{A.11}$$

Taking $\theta = \theta_n^-$ into (A.10) and passing to the limit, using the continuity of $p(\theta)$, yields

$$p(\theta_0)\dot{U}^*(\theta_0^-) - L_\Delta(U^*(\theta_0), \dot{U}^*(\theta_0^-)) \geq p(\theta_0)v - L_\Delta(U^*(\theta_0), v) \quad \forall\, v \leq 2\sqrt{-U^*(\theta)}.$$

Using similar arguments with the sequence θ_n^+, we also get

$$p(\theta_0)\dot{U}^*(\theta_0^+) - L_\Delta(U^*(\theta_0), \dot{U}^*(\theta_0^+)) \geq p(\theta_0)v - L_\Delta(U^*(\theta_0), v) \quad \forall\, v \leq 2\sqrt{-U^*(\theta)}.$$

Hence, the function $v \to p(\theta_0)v - L_\Delta(U^*(\theta_0), v)$ defined for $v \leq 2\sqrt{-U^*(\theta_0)}$ achieves its maxima at both $\dot{U}^*(\theta_0^+)$ and $\dot{U}^*(\theta_0^-)$. Since it is strictly concave, we get $\dot{U}^*(\theta_0^+) = \dot{U}^*(\theta_0^-)$. From this contradiction, we conclude that any arbitrary $\theta \in \Theta$ is contained in a relatively open interval on which $\dot{U}^*$ is almost everywhere equal to a continuous function. $\dot{U}^*$ and thus x_1^* and x_2^* are continuous on Θ. ■

Proof of Proposition 4. Since the functional $L_\Delta(\cdot)$ does not depend on θ, we can obtain a first quadrature of (12) on any interval where $U(\theta) + \frac{\dot{U}^2(\theta)}{4} < 0$ as:

$$L_\Delta(U(\theta,\lambda), \dot{U}(\theta,\lambda)) - \dot{U}(\theta,\lambda)\frac{\partial L_\Delta}{\partial \dot{U}}(U(\theta,\lambda), \dot{U}(\theta,\lambda)) = \lambda + \delta^2 + \frac{\Delta^2}{4}, \tag{A.12}$$

where a priori $\lambda \in \mathbb{R}$ and where we make explicit the dependence of the solution on this parameter. We obtain immediately:

$$U(\theta,\lambda) + \delta\dot{U}(\theta,\lambda) + \Delta\sqrt{-U(\theta,\lambda) - \frac{\dot{U}^2(\theta,\lambda)}{4}} - \dot{U}(\theta,\lambda)\left(\delta - \frac{\Delta\dot{U}(\theta,\lambda)}{4\sqrt{-U(\theta,\lambda) - \frac{\dot{U}^2(\theta,\lambda)}{4}}}\right) = -\lambda.$$

Simplifying yields:

$$U(\theta,\lambda)\left(1 - \frac{\Delta}{\sqrt{-U(\theta,\lambda) - \frac{\dot{U}^2(\theta,\lambda)}{4}}}\right) = -\lambda.$$

Solving for $\dot{U}(\theta,\lambda)$ yields

$$\dot{U}^2(\theta,\lambda) = -4\left(U(\theta,\lambda) + \Delta^2\left(\frac{U(\theta,\lambda)}{U(\theta,\lambda)+\lambda}\right)^2\right), \tag{A.13}$$

which requires $-\Delta^2\left(\frac{U(\theta,\lambda)}{U(\theta,\lambda)+\lambda}\right)^2 \geq U(\theta,\lambda)$ or $(U(\theta,\lambda)+\lambda)^2 + \Delta^2 U(\theta,\lambda) \geq 0$ given that $U(\theta,\lambda) \leq 0$ since by definition the agent's information rent is negative. Solving the second-order equation (A.13) and keeping the positive root only,[16] we get (14). When $\dot{U}(\theta,\lambda) > 0$, differentiating (14) with respect to θ yields

$$\ddot{U}(\theta,\lambda) + \frac{\dot{U}(\theta,\lambda)\left(1 + 2\lambda\Delta^2\frac{U(\theta,\lambda)}{(U(\theta,\lambda)+\lambda)^3}\right)}{\sqrt{-U(\theta,\lambda) - \Delta^2\left(\frac{U(\theta,\lambda)}{U(\theta,\lambda)+\lambda}\right)^2}} = \ddot{U}(\theta,\lambda) + 2\left(1 + 2\lambda\Delta^2\frac{U(\theta,\lambda)}{(U(\theta,\lambda)+\lambda)^3}\right) = 0 \tag{A.14}$$

Hence, on any interval where $\dot{U}(\theta,\lambda) > 0$, the second-order condition (9) can be written as

$$0 \leq \ddot{U}(\theta,\lambda) + 2 = -4\lambda\Delta^2\frac{U(\theta,\lambda)}{(U(\theta,\lambda)+\lambda)^3}. \tag{A.15}$$

Since $U(\theta,\lambda) \leq 0$ holds, $\lambda \leq 0$ implies also $U(\theta,\lambda) + \lambda \leq 0$ and then (A.15) holds. This imposes the requested restriction on the admissible solutions to (14). Finally, note that the second-order condition (9) holds obviously on any interval where instead $\dot{U}(\theta,\lambda) = 0$. ■

Proof of Theorem 1. The structure of the proof is as follows. First, we derive from the necessary free end-point conditions (13) some properties of the boundary values of U^* that are used to find λ^*. Sufficiency follows.

Necessity: Define the function $P(x) = \frac{-x((x+\lambda)^2+\Delta^2 x)}{(x+\lambda)^2}$. For $x < 0$, $P(x) > 0$ if and only if the second degree polynomial $(x+\lambda)^2 + \Delta^2 x$ is everywhere positive. This is so when $\lambda < -\frac{\Delta^2}{4}$. When that condition holds, the differential equation (14) is Lipschitz at any point where $U(\theta,\lambda) < 0$ and thus it has a single solution at any such point. Moreover, a solution $U(\theta,\lambda)$ is then everywhere increasing on the whole domain where $U(\theta,\lambda) < 0$. As a result, the differential equation (14) is everywhere Lipschitz when $U(1,\lambda) < 0$ which turns out to be the case for the path we derive below.

The necessary free end-points conditions (13) can be rewritten for an optimal path as:

$$\left(-\delta + \Delta\frac{\dot{U}^*(\theta)}{4\sqrt{-U^*(\theta) - \frac{(\dot{U}^*(\theta))^2}{4}}}\right)|_{\theta=0,1} = 0.$$

Using (14) to express $\dot{U}^*(\theta)$, those conditions can be simplified so that $U^*(0)$ and $U^*(1)$ solve indeed the following second-order equation in U:

$$(U+\lambda^*)^2 = -(\Delta^2 + 4\delta^2)U, \tag{A.16}$$

[16] Since it corresponds to an average decision $x(\theta)$ biased towards the principal, namely $x(\theta) \geq \theta$ (see equation (8)).

where λ^* is the value of λ for the optimal arc U^*. Assuming now that $\lambda^* > -\frac{\Delta^2}{4} - \delta^2$ (a condition checked below), (A.16) admits two solutions respectively given by

$$U^*(0) = -\lambda^* - \frac{1}{2}\left(\Delta^2 + 4\delta^2 + \sqrt{(\Delta^2 + 4\delta^2)^2 + 4\lambda^*(\Delta^2 + 4\delta^2)}\right), \tag{A.17}$$

$$U^*(1) = -\lambda^* - \frac{1}{2}\left(\Delta^2 + 4\delta^2 - \sqrt{(\Delta^2 + 4\delta^2)^2 + 4\lambda^*(\Delta^2 + 4\delta^2)}\right). \tag{A.18}$$

Note in particular that (A.16) implies that both solutions are negative.

The last step is to show that there exits $\lambda^* \in (-\frac{\Delta^2}{4} - \delta^2, -\frac{\Delta^2}{4})$ such that the corresponding path $U^*(\theta) = U(\theta, \lambda^*)$ solving (14) and starting from

$$U(0, \lambda) = -\lambda - \frac{1}{2}\left(\Delta^2 + 4\delta^2 + \sqrt{(\Delta^2 + 4\delta^2)^2 + 4\lambda(\Delta^2 + 4\delta^2)}\right),$$

reaches

$$U(1, \lambda) = -\lambda - \frac{1}{2}\left(\Delta^2 + 4\delta^2 - \sqrt{(\Delta^2 + 4\delta^2)^2 + 4\lambda(\Delta^2 + 4\delta^2)}\right).$$

This requires to find a solution λ^* to the equation $\varphi(\lambda) = \psi(\lambda)$, with

$$\varphi(\lambda) = U(1, \lambda) - U(0, \lambda) = \sqrt{(\Delta^2 + 4\delta^2)^2 + 4\lambda(\Delta^2 + 4\delta^2)},$$

and

$$\psi(\lambda) = \int_0^1 \dot{U}(\theta, \lambda) d\theta = \int_0^1 2\sqrt{-U(\theta, \lambda) - \Delta^2 \left(\frac{U(\theta, \lambda)}{U(\theta, \lambda) + \lambda}\right)^2} d\theta,$$

where the path $U(\theta, \lambda)$ starts from the initial condition $U(0, \lambda)$. Note that both $\varphi(\cdot)$ and $\psi(\cdot)$ are continuous in λ. It is clear that $\varphi(\lambda)$ is strictly increasing in λ with, for $\lambda_1 = -\frac{\Delta^2}{4} - \delta^2$ and $\lambda_2 = -\frac{\Delta^2}{4}$,

$$\varphi(\lambda_1) = 0 < 2\delta\sqrt{\Delta^2 + 4\delta^2} = \varphi(\lambda_2).$$

On the other hand, note that

$$\psi(\lambda_1) > 0 = \varphi(\lambda_1), \tag{A.19}$$

since the path $U(\theta, \lambda_1)$ starting from $U(0, \lambda_1)$ is strictly increasing. Moreover, for λ_2, (14) can be rewritten as:

$$\dot{U}(\theta, \lambda_2) = 2\sqrt{-U(\theta, \lambda_2)} \frac{|U(\theta, \lambda_2) - \lambda_2|}{|U(\theta, \lambda_2) + \lambda_2|}. \tag{A.20}$$

The path solving (A.20) and starting at $U(0, \lambda_2)$ (note that $U(0, \lambda_2) < \lambda_2 < U(1, \lambda_2)$) is strictly increasing everywhere and cannot cross the boundary $U = \lambda_2$ because the only solution to (A.20) such that $U(\theta_1) = \lambda_2$ for a given $\theta_1 > 0$ is such that $U(\theta) = \lambda_2$ for all θ since the right-hand side of (A.20) satisfies a Lipschitz condition at any point $U(\theta, \lambda_2)$ away from zero; a contradiction with $U(0, \lambda_2) < \lambda_2$. From that, we deduce $U(\theta, \lambda_2) < \lambda_2$ for all θ. Hence, the following sequence of inequalities holds:

$$\psi(\lambda_2) = \int_0^1 \dot{U}(\theta, \lambda_2) d\theta < \lambda_2 - U(0, \lambda_2) = \lambda_2 - U(1, \lambda_2) + \varphi(\lambda_2).$$

Finally, we get:

$$\psi(\lambda_2) < \varphi(\lambda_2). \tag{A.21}$$

Gathering Equations (A.19) and (A.21) yields the existence of $\lambda^* \in (\lambda_1, \lambda_2)$ such that $\varphi(\lambda^*) = \psi(\lambda^*)$.

Sufficiency: Sufficiency follows from Clarke (1990, Chapter 4, Corollary p. 179) when noticing that $L_\Delta(\cdot)$ satisfies the convexity assumption and the function $s \to H(s, p(\theta))$ is concave in s. ■

Proof of Corollary 1. First observe that

$$x^*(0) = \frac{\dot{U}^*(0)}{2} = \frac{\sqrt{-U^*(0)((U^*(0)+\lambda^*)^2 + \Delta^2 U^*(0))}}{|U^*(0)+\lambda^*|}.$$

Using (A.17), we get:

$$x^*(0) = 2\delta \frac{|U^*(0)|}{|U^*(0)+\lambda^*|} < 2\delta.$$

Finally, we obviously have $x^*(\theta) - \theta = \frac{\dot{U}^*(\theta)}{2} > 0$ for all θ. ■

Proof of Corollary 2. The first property follows from (16). Now, from the incentive constraint (3) and the first property of the corollary we get the second property. Next, using (16) we have $x_1^*(0) \geq 0$ if and only if

$$\frac{\dot{U}^*(0)}{2} \geq \Delta\left(\frac{U^*(0)}{U^*(0)+\lambda^*}\right).$$

Using (14) and simplifying we get $2\Delta^2 U^*(0) \geq -(U^*(0)+\lambda^*)^2$, i.e., by (A.16), $2\Delta^2 \leq \Delta^2 + 4\delta^2$, which is always satisfied (with a strict inequality when $\delta_1 > 0$). $\dot{x}_2^*(0) \leq 0$ follows now from the second property of the corollary. The last property is proved similarly. ■

Proof of Proposition 5. We have

$$\int_0^1 L_\Delta(U^*(\theta), \dot{U}^*(\theta))d\theta \geq \int_0^1 \left(L_0(U^*(\theta), \dot{U}^*(\theta)) - \Delta\sqrt{-U^*(\theta) - \frac{\left(\dot{U}^*(\theta)\right)^2}{4} + \frac{\Delta^2}{4}}\right) d\theta$$

$$\geq \int_0^1 \left(L_0(U_0^*(\theta), \dot{U}_0^*(\theta)) - \Delta\sqrt{-U^*(\theta) - \frac{\left[\dot{U}^*(\theta)\right]^2}{4} + \frac{\Delta^2}{4}}\right) d\theta$$

$$= \delta^2 + \frac{\Delta^2}{4} - \frac{4\delta^3}{3} - \Delta^2 \int_0^1 \frac{|U^*(\theta)|}{|U^*(\theta)+\lambda^*|} d\theta,$$

where $U_0^*(\theta)$ is given in (17) and the last inequality follows from (14). This implies:

$$\int_0^1 L_\Delta(U^*(\theta), \dot{U}^*(\theta))d\theta \geq \delta^2 - \frac{3\Delta^2}{4} - \frac{4\delta^3}{3}.$$

We also have

$$\int_0^1 L_\Delta(\tilde{U}^*(\theta), \dot{\tilde{U}}^*(\theta))d\theta = \delta^2 - \frac{4\delta^3}{3},$$

and

$$\int_0^1 L_\Delta(U^O(\theta), \dot{U}^O(\theta))d\theta = \delta^2 + \frac{\Delta^2}{4} - \frac{2\delta_1^3}{3} - \frac{2\delta_2^3}{3},$$

which gives the required inequalities. ■

Proof of Proposition 6. A stochastic direct mechanism is a mapping $\mu(\cdot|\cdot) : \Theta \to \Delta(\mathcal{K} \times \mathcal{K})$ where $\Delta(\mathcal{K} \times \mathcal{K})$ is the set of measures on $\mathcal{K} \times \mathcal{K}$. For further references, we define the mean and variance of such stochastic mechanism as

$$\bar{x}_i(\hat{\theta}) = \int_{\mathcal{K}\times\mathcal{K}} x_i d\mu(x_1, x_2|\hat{\theta}) \text{ and } \sigma_i^2(\hat{\theta}) = \int_{\mathcal{K}\times\mathcal{K}} (x_i - \bar{x}_i(\hat{\theta}))^2 d\mu(x_1, x_2|\hat{\theta}) \geq 0.$$

Boundedness of $\mathcal{K}$ ensures that such moments exist. Note that deterministic mechanisms are such that $\sigma_i^2(\hat{\theta}) \equiv 0$. For further references also, denote $\bar{x}(\hat{\theta}) = \frac{1}{2}\sum_{i=1}^2 \bar{x}_i(\hat{\theta})$ the average decision and $\bar{y}(\hat{\theta}) = \bar{x}_2(\hat{\theta}) - \bar{x}_1(\hat{\theta})$ the spread of those average decisions. In this context, incentive compatibility can be written as:

$$U(\theta) = \max_{\hat{\theta}\in\Theta} \int_{\mathcal{K}\times\mathcal{K}} \left(\sum_{i=1}^{2} -\frac{1}{2}(x_i - \theta)^2\right) d\mu(x_1, x_2|\hat{\theta}).$$

Taking expectations, we get:

$$U(\theta) = \max_{\hat{\theta}\in\Theta} -\frac{1}{2}\left(\sum_{i=1}^{2}(\bar{x}_i(\hat{\theta}) - \theta)^2\right) - z(\hat{\theta}) = \max_{\hat{\theta}\in\Theta} -(\bar{x}(\hat{\theta}) - \theta)^2 - \frac{\bar{y}^2(\hat{\theta})}{4} - z(\hat{\theta}),$$

where $z(\hat{\theta}) = \frac{1}{2}\sum_{i=1}^2 \sigma_i^2(\hat{\theta}) \geq 0$. From this, it immediately follows that $U(\cdot)$ is absolutely continuous with a derivative defined almost everywhere defined as

$$\dot{U}(\theta) = 2(\bar{x}(\hat{\theta}) - \theta), \tag{A.22}$$

with

$$z(\theta) \leq -U(\theta) - \frac{\dot{U}^2(\theta)}{4} - \frac{\bar{y}^2(\theta)}{4}. \tag{A.23}$$

Similarly, the expected payoff of the principal with such a stochastic mechanism can be written as:

$$\int_0^1 \left(\int_{\mathcal{K}\times\mathcal{K}} \left(\sum_{i=1}^{2} -\frac{1}{2}(x_i - \theta - \delta_i)^2\right) d\mu(x_1, x_2|\theta)\right) d\theta = \int_0^1 \left((U(\theta) + \delta\dot{U}(\theta) + \frac{\Delta}{2}\bar{y}(\theta) - \delta^2 - \frac{\Delta^2}{4}\right) d\theta.$$

The principal problem when stochastic mechanisms are allowed can be written as:

$$(\mathcal{P}^s_\Delta) : \quad \min_{\{U\in W^{1,1}(\Theta), z\geq 0\}} \int_0^1 L^s_\Delta(U(\theta), \dot{U}(\theta), z(\theta))d\theta,$$

where

$$L^s_\Delta(U(\theta), \dot{U}(\theta), z(\theta)) = -U(\theta) - \delta\dot{U}(\theta) - \Delta\sqrt{-U(\theta) - \frac{\dot{U}^2(\theta)}{4} - z(\theta)} + \delta^2 + \frac{\Delta^2}{4}.$$

Clearly, the pointwise solution to this problem when $\Delta > 0$ is achieved for $z(\theta) = 0$, i.e., for deterministic mechanisms. ■

Proof of Proposition 7. Suppose that the optimal solution U^* is such that $U^*(\theta) = 0$ on an interval I with non-empty interior, i.e., $x_i^*(\theta) = \theta$ on that interval. Consider now the new utile profile $\tilde{U}$ obtained by leaving the decisions $x_i^*(\theta)$ unchanged on I^c but choosing $\tilde{x}_1(\theta) = \theta - \epsilon$ and $\tilde{x}_2(\theta) = \theta + \epsilon$ on I. Note that $\dot{\tilde{U}}(\theta) = \dot{U}^*(\theta)$ both on I and I^c. Observe that

$$\int_0^1 L_\Delta(\tilde{U}(\theta), \dot{\tilde{U}}(\theta))f(\theta)d\theta - \int_0^1 L_\Delta(U^*(\theta), \dot{U}^*(\theta))f(\theta)d\theta$$

$$= \int_I (L_\Delta(\tilde{U}(\theta), \dot{\tilde{U}}(\theta)) - L_\Delta(U^*(\theta), \dot{U}^*(\theta)))f(\theta)d\theta = -(\epsilon\Delta - \epsilon^2)\int_I f(\theta)d\theta < 0,$$

for ϵ small enough, a contradiction with the optimality of U^*. ■

References

Alonso, N. and R. Matouschek (2007): "Relational Delegation," *Rand Journal of Economics*, 38, 70–89.

——— (2008): "Optimal Delegation," *Review of Economic Studies*, 75, 259–293.

Alonso, R., W. Dessein, and N. Matouschek (2008): "When Does Coordination Require Centralization?" *American Economic Review*, 98, 145–179.

Ambrus, A. and S. Takahashi (2008): "Multi-Sender Cheap Talk with Restricted State Space," *Theoretical Economics*, 3, 1–27.

Armstrong, M. (1994): "Delegation and Discretion," Discussion Papers in Economics and Econometrics 9421, University of Southampton.

Arnott, R. and J. Stiglitz (1988): "Randomization with Asymmetric Information," *Rand Journal of Economics*, 19, 344–362.

Athey, S., A. Atkeson, and P. Kehoe (2005): "Optimal Degree of Discretion in Monetary Policy," *Econometrica*, 73, 1431–1475.

Austen-Smith, D. (1993): "Interested Experts and Policy Advice: Multiple Referrals under Open Rule," *Games and Economic Behavior*, 5, 3–44.

Baron, D. (2000): "Legislative Organization with Informational Committees," *American Journal of Political Sciences*, 44, 485–505.

Battaglini, M. (2002): "Multiple Referrals and Multidimensional Cheap Talk," *Econometrica*, 70, 1379–1401.

——— (2004): "Policy Advice with Imperfectly Informed Experts," *Advances in Theoretical Economics*, 4.

Clarke, F. H. (1990): *Optimization and Nonsmooth Analysis*, SIAM. Philadelphia.

Crawford, V. P. and J. Sobel (1982): "Strategic Information Transmission," *Econometrica*, 50, 1431–1451.

Dessein, W. (2002): "Authority and Communication in Organizations," *Review of Economic Studies*, 69, 811–832.

Epstein, D. and S. O'Halloran (1999): *Delegating Powers*, Cambridge University Press.

Garcia, D. (2005): "Monotonicity in Direct Revelation Mechanisms," *Economics Letters*, 88, 21–26.

Gilligan, T. W. and K. Krehbiel (1987): "Collective Decision-Making and Standing Committees: An Informational Rationale for Restricting Amendment Procedures," *Journal of Law, Economics, and Organization*, 3, 287–335.

Goltsman, M., J. Hörner, G. Pavlov, and F. Squintani (2007): "Mediation, Arbitration and Negotiation," mimeo, University of Western Ontario.

Harris, M. and A. Raviv (1996): "The Capital Budgeting Process, Incentives and Information," *Journal of Finance*, 51, 1139–1174.

Holmström, B. (1984): "On the Theory of Delegation," in *Bayesian Models in Economic Theory*, ed. by M. Boyer and R. Kihlstrom, Elsevier Science B. V.

Huber, J. and C. Shipan (2002): *Deliberative Discretion: The Institutional Foundations of Bureaucratic Autonomy*, Cambridge University Press.

——— (2006): "Politics, Delegation, and Bureaucracy," in *The Oxford Handbook of Political Economy*, ed. by B. Weingast and D. Wittman, Oxford University Press.

Kovac, E. and T. Mylovanov (2009): "Stochastic Mechanisms in Settings without Monetary Transfers: The Regular Case," *Journal of Economic Theory*, 144, 1373–1395.

Krishna, V. and J. Morgan (2001): "A Model of Expertise," *Quarterly Journal of Economics*, 116, 747–775.

Laffont, J.-J. and D. Martimort (2002): *The Theory of Incentives: The Principal-Agent Model*, Princeton University Press.

Levy, G. and R. Razin (2007): "On the Limits of Communication in Multidimensional Cheap Talk: A comment," *Econometrica*, 75, 885–894.

Lewis, T. R. and D. E. M. Sappington (1989a): "Countervailing Incentives in Agency Problems," *Journal of Economic Theory*, 49, 294–313.

——— (1989b): "Inflexible Rules in Incentive Problems," *American Economic Review*, 79, 69–84.

Martimort, D. and A. Semenov (2006): "Continuity in Mechanism Design without Transfers," *Economic Letters*, 93, 182–189.

——— (2008): "The Informational Effects of Competition and Collusion in Legislative Politics," *Journal of Public Economics*, 92, 1541–1563.

McCubbins, M., R. Noll, and B. Weingast (1987): "Administrative Procedures as Instruments of Political Control," *The Journal of Law, Economics and Organization*, 3, 243–277.

Melumad, N. D. and T. Shibano (1991): "Communication in Settings with no Transfers," *Rand Journal of Economics*, 22, 173–198.

Milgrom, P. and I. Segal (2002): "Envelope Theorems for Arbitrary Choice Sets," *Econometrica*, 70, 583–601.

Mirrlees, J. A. (1971): "An Exploration in the Theory of Optimum Income Taxation," *Review of Economic Studies*, 38, 175–208.

Myerson, R. B. (1982): "Optimal Coordination Mechanisms in Generalized Principal-Agent Problems," *Journal of Mathematical Economics*, 10, 67–81.

Royden, H. (1988): *Real Analysis*, 3rd Edition, Prentice Hall.