Optimal Collusion with Limited Liability and Policy Implications *

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Optimal Collusion with Limited Liability and Policy Implications

Abstract

Collusion sustainability depends on firms' aptitude to impose sufficiently severe punishments in case of deviation from the collusive rule. We extend results from the literature on optimal collusion by investigating the role of limited liability. We examine all situations in which either structural conditions (demand and technology), financial considerations (a profitability target), or institutional circumstances (a regulation) set a lower bound, possibly negative, to firms' profits. For a large class of repeated games with discounting, we show that, absent participation and limited liability constraints, there exists a unique optimal penal code. It commands a severe single-period punishment immediately after a firm deviates from the collusive stage-game strategy. When either the participation constraint or the limited liability constraint bind, there exists an infinity of multi-period optimal punishment paths that permit firms to implement the collusive strategy. The usual front-loading scheme is only a specific case, an optimal punishment profile can take the form of a price asymmetric cycle, and a longer punishment is not always a perfect substitute for more immediate severity. Limiting further firms' liability may actually result in more efficient markets and can be used as a regulatory instrument. Absent collusion, introducing a price floor slightly *below* the observed transaction price has no impact on firms' behavior. However, the floor makes collusive equilibria unsustainable.

JEL Classification: C72, D43, L13

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1 Introduction

In this paper, we characterize the ability of oligopolistic firms to implement a collusive strategy when their ability to punish deviations over one or several periods is limited.

Firms in the same industry may increase profits by coordinating the prices they charge or the quantities they sell. In a legal context in which collusive agreements cannot be overtly enforced, and future profits are discounted, it is well-known that an impatient firm may find it privately profitable to deviate from a collusive strategy. This renders collusive agreements fundamentally unstable. However, firms may design non-cooperative discipline mechanisms that help implementing collusion.

Many papers examine the structural conditions that facilitate the formation of cartels. Most theoretical analyses rely on a class of dynamic models usually referred to as supergames. These models feature a repeated market game in which firms maximize a flow of discounted individual profits by non-cooperatively choosing a price or a quantity over an infinite number of periods. When a deviation can be credibly and sufficiently "punished" via lower industry prices or larger quantities in subsequent time periods, conditions on structural parameters can be derived which, when satisfied, make collusion stable.

A majority of recent contributions to the literature investigate the impact of various model specifications on the sustainability of collusion with stick-and-carrot mechanisms in the style of Abreu (1986, 1988). In this category of mechanisms, if a firm deviates from collusion, all firms play a punishment strategy over one or several periods – the stick – which is more severe than Nash reversion (i.e., it leads to lower instantaneous profits, possibly negative) before returning to a collusive price or quantity. If a deviation occurs in a punishment period, the punishment phase restarts, otherwise all firms resume the collusive behavior to earn supernormal profits – the carrot. More specifically, Abreu (1986) exploits a single-period punishment mechanism for a class of repeated quantity-setting oligopoly stage games with symmetric sellers of a homogenous good, constant positive marginal costs, and no fixed cost. For a given discount factor, the most severe punishment strategy – following a deviation either from the collusive path or from a punishment rule – that sustains collusion, is characterized. It results in the highest level of discounted collusive profits.

The analysis of the connection between structural conditions and collusion stability with a stick-and-carrot mechanism *à la* Abreu has been extended to many aspects. They include the case of multi-market contact (Bernheim and Whinston (1990)). Collusion is facilitated when the

same firms are present on several markets. Capacity constraints have been considered (in particular Lambson (1987, 1994), Compte, Jenny, Rey (2002)). A general message is that limited and asymmetric capacities make collusion more difficult to sustain. Other papers focus on cost heterogeneity (including Rothschild (1999), Vasconcelos (2005), Miklós-Thal (2011)). It is found that collusion is more difficult to sustain when costs are asymmetric, and that collusion sustainability depends on the difference between the marginal cost levels that characterize both the less and the most efficient firms in the industry. Another research stream focuses on circumstances in which each firm receives a cost shock in each period of a repeated price-setting game with infinite horizon (notably Athey et al. (2001, 2004, 2008)). An important result is that, when marginal costs are private information and may differ across firms, and under simple and general assumptions, ex ante cartel payoffs are maximized when firms charge the same collusive price and share the market equally, as in simpler models with complete information and symmetric firms. Other contributions, which do not always allow for the possibility of pricing below marginal costs, investigate the impact of changes in demand, with various specifications for the dynamics of shocks (see, in particular, Rotemberg and Saloner (1986), Haltiwanger and Harrington (1991), Bagwell and Staiger (1997)). A "tuned" collusive price gets closer to the competitive level when demand is high.[1]

Our objective is to enrich the study of the circumstances that facilitate collusion, or make it more difficult to sustain. This is done by investigating the exact role of an assumption, in the seminal paper by Abreu (1986), according to which the price is *strictly* positive for all levels of industry output, so that there is no floor for firms' losses when the constant marginal cost is also specified above zero. Indeed the quantity sold – and related costs – tend to infinity when firms charge below the marginal cost and the price approaches zero. In that case, the *single-period* punishment that follows a deviation can be made *as severe as needed*. Although the strategy set is assumed to be finite, the upper bound to the available quantities is so high as to never be used as a punishment action that sustains collusion.

To our knowledge, most papers – if not all – that refer to Abreu (1986, 1988) actually overlook this key assumption by introducing more structure. They typically borrow the same stick-and-carrot mechanism with a single punishment period, although they either assume that demand is finite at all prices, or that firms have limited production capacity. It follows that losses are bounded from below in a punishment period, and collusion can be hindered. In that case, an extension of the punishment phase to several periods appears as a natural substitute for more immediate severity. Fudenberg and Tirole (1991, p. 165) emphasize that, when the severity of punishments

[1] For a comprehensive survey of the literature on the factors that facilitate collusion, see Motta (2004).

is limited the punishment phase should be longer, although "it is not obvious precisely which actions should be specified" in the punishment phase. Our paper is novel in that it thoroughly examines this point. This is done in a setup that encompasses the main assumptions in Abreu (1986). In our model, firms sell substitutable goods (possibly differentiated), inverse demand functions are non-increasing (they can be finite at all prices), the marginal cost is constant and non-negative (it can be zero), and there can be a fixed cost. In addition to standard incentive and participation constraints, a key specification that we introduce is the limited liability constraint, which amounts to imposing a limitation on the lowest level of profits a firm may earn. Whether the limited liability constraint binds or not impacts firms' choices of price or quantity in the punishment phase.

Interestingly, a limited liability constraint is not a technical sophistication that we add to standard specifications. It is *de facto* present, or latent, in all models where demand or technological conditions set a lower bound to firms' losses. A finite demand, or a limited capacity, are examples of structural specifications that constrain firms' payoffs to remain above a certain (non-positive) level. Then, firms' losses also remain finite when the prices they charge are below their unit costs of production. This limits the maximal severity of punishment schemes. In this case, a firm with high fixed and/or variable costs earns more negative payoffs during aggressive pricing episodes than more efficient firms. This offers a new explanation for an empirical observation by Symeonidis (2003), who finds strong evidence that collusion is more likely in industries with high capital intensity. This result has been interpreted as a consequence of high barriers to entry.[2] Another possible and more direct interpretation, which we investigate below, is that high average costs – which permit severe punishments – facilitate collusion.

To illustrate, consider for example a linear and symmetric n-firm Cournot oligopoly.[3] The non negative unit cost of production c is constant, and firm i's inverse demand function in each period is the non negative part of $p_i(q_i, q_j) = 1 - q_i - \gamma(n-1)q_j$, where p_i is the price it charges, q_i is the quantity it sells, q_j is the quantity sold by each competitor, and γ is a measure of product substitutability. In this standard model, the limited liability constraint is not a specification we add. It is already rooted in the usual assumption that the quantity demanded is finite at all prices, which cannot be negative. Hence the severity of punishments is limited. We establish that the limitation plays no role for all admissible values of c and γ only if there are exactly two or three firms. In that case, the results obtained in the related literature with a duopoly

[2]In Symeonidis (2003), the capital stock of the average plant, and the capital-labor ratio, are proxies for high barriers to entry, which in turn are seen to facilitate collusion. See also Levenstein and Suslow (2006).

[3]This example is studied extensively in Section 5.

and a constant marginal (and unit) cost normalized to zero are robust to the introduction of a positive marginal cost. In all other cases, that is, with more than three firms, the limited liability constraint binds for some values of the cost and differentiation parameters. This was not found in past theoretical studies, although the intuition for that result appears in an exploratory note by Lambertini and Sasaki (2001, p. 119), who explain that "high marginal costs tend to provide more room for tacit collusion than ... with lower marginal costs, due to the positive price constraint." Moreover, introducing a positive fixed cost reduces the parameter subset in which the limited liability constraint binds; therefore, a high unit cost facilitates collusion. When the limited liability constraint binds, we establish that the lowest discount value for which collusion can be implemented decreases when the number of firms decreases, and either differentiation or the marginal cost increases. This extends existing results to situations in which there is a limited liability constraint, and it also emphasizes that all factors enhancing the firms' ability to punish – in that they relax the limited liability constraint – facilitate collusion.

It is also well-known that financial parameters (e.g., a return on investment target) may also shape the limited liability constraint. For example, prudential ratios set a limit to the quantity of loans a bank may supply. Another example is that financial markets constrain managers of equity-dependent firms not to post low operational profits for too long. The empirical literature has evidenced the connection between stock prices and firms' investments, as in Baker et al. (2003). Our theoretical analysis establishes that there is also a link between financial constraints and the ability to collude.

Finally, the limited liability constraint can capture all real-world contexts in which institutional circumstances (e.g., regulation) impact firms' behavior. An example of a regulatory measure that reduces the severity of punishments is a price floor. As it rules out severe punishments, it should hinder collusion. In an empirical paper, Gagné et al. (2006) study the impact on prices of a price floor established by the Quebec provincial government on the retail market for gasoline. By limiting the severity of price wars, the floor was seen as a means to reduce the ability of firms to punish retailers deviating from a high price strategy. The analysis reveals that the net effect of the floor on average price-cost margins is near zero. The impact of the floor on retail prices in low margin periods (or price wars) is actually offset by the rise in their average duration. Price wars are less severe, but they last longer.[4] Our analysis offers theoretical grounds to these empirical

[4]The introduction of a price floor followed a price war. The local association of independent gasoline retailers reported that the price war "resulted in retail prices that were observed well below wholesale prices. It was so severe as to force several independent retailers either to close down temporarily or to exit the market" (translated from the *Mémoire de l'Association Québécoise des Indépendants du Pétrole*, June 1998, pp. 7-8). In another empirical

findings.

In this paper, by delineating the largest parameter space for which a collusive strategy can be implemented, we fully characterize the conditions under which the limited liability constraint does reduce the firms' ability to implement a given collusive action (a price or a quantity), in a large class of models where the duration of punishments can be adjusted. For given cost and demand parameters, the optimal punishment path is defined as a vector of prices or quantities, played period after period, that let firms implement a given collusive strategy for the lowest admissible discount factor. When only incentive constraints are at play, there is a *unique* optimal punishment path.

When the limited liability constraint is slack, we find that the possibility to punish over several periods does *not* result in a lower threshold for the discount factor than with a single-period punishment scheme that we use as a benchmark. This also holds with a binding participation constraint. The latter specifies a minimum continuation payoff following a deviation, but says nothing on the distribution of this payoff over time.

When the limited liability constraint binds, we find that there exists an infinity of simple punishment paths that permit firms to implement the collusive strategy. The lowest discount factor for which a given collusive strategy can be implemented strictly decreases if the punishment phase is not limited to a single period. We establish that this discount threshold is always reached with a punishment phase of *finite* length. Only in particular circumstances, which we characterize, the discount threshold is as low as in the case without the limited liability constraint. In all other cases, the discount threshold remains *strictly* higher than in the absence of a limited liability constraint. This means that, although the duration of the punishment phase is not bounded, the limited liability constraint still handicaps the firms' ability to collude. In other words, a longer punishment with discounting offers only an imperfect substitute for more immediate severity.

At a theoretical level, the main lesson of the paper is thus that models of collusion, when they do not take into account the (latent) limited liability constraint, usually exaggerate the sustainability of collusive agreements. At the policy level, the implication is straightforward, although it goes against common wisdom. In the context of our model, all attempts that amount to limiting further the firms' liability improve market efficiency. Depending on the circumstances under scrutiny, this can take the form of a cost reduction, better control of predatory pricing, or a higher profitability target.

analysis of the impact of this regulation, Houde (2008) finds that the minimum retail price floor had a significant impact on the firms' option value of staying in the market.

More specifically, we exploit our theoretical results to design a very simple regulatory mechanism which uses only *prima facie* observable information to restore competition. The regulation introduces a price floor slightly *below* the observed transaction price, and thereby merely rules out extreme variations in prices. If firms do not collude, the constraint simply does not impact the firms' non-collusive actions. Only in case of collusion, the constraint necessarily induces a change in firms' behavior. It allows the regulator to drive the industry to the stage-game Nash equilibrium by iteration.

The remainder of the paper is organized as follows. Section 2 describes the model. In Section 3, we restrict the duration of a punishment phase to a single period and identify the largest space of parameters for which a collusive strategy can be implemented. In Section 4, we obtain the main results by investigating the impact of punishing over several periods on the firms' ability to collude. In section 5, the latter results are illustrated in the context of a linear Cournot model. In Section 6, we discuss our results in the light of the related literature. Finally, in Section 7 we construct the regulatory mechanism. Intermediate results, and detailed proofs, are relegated to the appendix.

2 The Model

We construct a supergame, in which symmetric firms in $N = \{1, \ldots, n\}$ supply substitutable goods, possibly differentiated, to maximize individual intertemporal profits by simultaneously and non-cooperatively choosing a strategy a_i – or "action" – that is either a price or a quantity in an infinitely repeated stage game over $t = 1, 2, ..., \infty$. Each firm's action set A is an interval of R_+. The discount factor $\delta = 1/(1 + r)$, where r is the single-period interest rate, is common to all firms. The continuous function $\pi_i : R_+^2 \to R_+$ relates firm i's profits to a vector of actions $\mathbf{a} \equiv (a_i, a_{-i})$, where a_{-i} describes a symmetric action chosen by all firms in $N\backslash\{i\}$. We omit the subscript i and specify a single argument a, which is a scalar, to represent the profits $\pi(a)$ earned by firms that all choose the same action. Similarly, we denote by $\pi_i^d(a)$ the profits firm i earns when it "deviates", in that it plays its best reply to a, as played by all other firms. The set of available actions includes a unique symmetric Nash equilibrium in pure strategy a_{NE}, implicitly defined by $\pi_i^d(a_{NE}) - \pi(a_{NE}) = 0$, all i, and a collusive action, a_m, which yields more profits (when $a_m = a_m^*$, it maximizes joint profits, a case of "perfect" collusion, as in the example we present in Section 5). Firms' actions may differ from period to period. An action path $\{\mathbf{a}^t\}_{t=1}^{\infty}$ is defined as an infinite stream of n-dimensional vectors of actions, as chosen by each firm in each period.

We give more structure to the analysis by relating each firm i's profits $\pi_i = p_i q_i - C(q_i)$, where p_i is a price q_i a quantity, to the exact properties of cost and demand conditions. There are three basic assumptions:

($A1$) Firms incur a fixed cost $f \geq 0$, and a variable cost $c(q_i) \geq 0$, to sell substitutable goods (possibly differentiated), and their strategic variable is either a (non-negative) price ($a = p$ in the Bertrand specification) or quantity ($a = q$ in the Cournot specification).

($A2$) Firm i's inverse demand function $p_i : R_+^n \rightarrow R_+$ is non-increasing and continuous.

($A3$) $p_i(\mathbf{0}) > c$ and $\lim_{q_i \rightarrow \infty} p_i(q_i, \mathbf{q}_{-i}) = 0$, any $\mathbf{q}_{-i}$ in R_+^{n-1}.

The main features of our model appear clearly when compared with the specifications in Abreu (1986), a reference, where the following three assumptions hold: ($\widetilde{A}1$) Firms sell a homogeneous good at constant marginal cost $c > 0$, and their strategic variable is quantity; ($\widetilde{A}2$) The market inverse demand function $p(q) : R_+ \rightarrow R_+$ is *strictly* decreasing and continuous in $q = \sum_{i \in N} q_i$; and ($\widetilde{A}3$) $p(0) > c$ and $\lim_{q \rightarrow \infty} p(q) = 0$. Note that the latter two assumptions imply that, for all levels of total output q, the price p is *strictly* positive. They also imply that there exists $q_c > 0$ such that $p(q_c) < c$. This says that firms can always force the price p at which firm i sells q_i down to a level strictly below c. In this case there is no floor for firms' losses since the quantity sold – and related costs – can tend to infinity when p approaches 0. The latter three assumptions are encompassed by ($A1$-$A3$). Note in particular that our assumptions also capture circumstances in which the price p_i is driven down to *exactly* zero with *finite* quantities $(q_i, \mathbf{q}_{-i})$, a case *ruled out* by Abreu's assumptions ($\widetilde{A}1$-$\widetilde{A}3$).[5]

As in Abreu (1986) we construct a "stick-and-carrot" penal code. All firms initially collude by choosing the collusive action a_m. If this action is played by all firms in all periods, each firm earns the discounted sum of the single-period (positive) collusive profits $\pi_m \equiv \pi(a_m)$. All firms have a short-run incentive to deviate, that is to lower (increase) its own price (quantity) in order to increase individual profits at every other firm's expense. If such a deviation is detected in period t, all firms switch to the punishment action a_P, in period $t+1$ (the stick). The choice

[5] In Abreu (1986, Assumption ($A4$), p. 195) each firm's strategy set is defined on a finite interval of quantities $S_i = [0, \bar{q}(\delta)]$, where $\bar{q}(\delta)$ satisfies $\pi_i(\bar{q}(\delta), 0) < -\frac{\delta}{1-\delta} \sup_{q_i} \pi_i(q_i, 0)$, in our notation. This means that $\bar{q}(\delta)$ is specified to be greater than the quantity a firm should sell to incur a loss equal in magnitude to the continuation profits, computed from the next period onward, it would earn as a monopolist in all periods forever. This upper bound in fact is so high as to be always greater than the single-period punishment quantity that sustains optimal collusion (see proof of Lemma 8, p. 201).

of a low (high) punishment price (quantity) a_P renders a free-riding behavior less attractive. If any deviation from a_P is detected, the punishment phase restarts, otherwise all firms resume the collusive behavior by adopting the same a_m forever (the carrot).

In order to express results and related proofs with notational parsimony, independently of the price and quantity specifications, hereafter we adopt the definition that the action a' is *more severe to firm i than* (strictly less severe than) a when $\pi(a') \leq (>)\pi(a)$. This is denoted by $a' \preceq_i (\succ_i) a$, where the subscript is omitted whenever no ambiguity is likely to result.

A key feature of the paper is that we investigate the consequence of having a lower bound to individual punishment actions, and thereby to punishment profits. We refer to this lower bound $\underline{a}_P \preceq_i a_{NE}$, for all i, as the most severe symmetric punishment action, a parameter. Given $\underline{a}_P$, we define $\underline{\pi} \equiv \pi(\underline{a}_P) \leq \pi(a_{NE})$. Most realistic circumstances offer a justification for this setting. It can capture the impact of a regulatory measure. For example, a price floor will impose firms to charge above a given value (say, a wholesale price), and then will limit the severity of punishment actions (in some cases we may have $\underline{\pi} > 0$). More generally, the severity of punishments is also limited when the demanded quantity is finite at any price, including zero, for all firms.[6] As indicated above, there is no such constraining limit on punishments in Abreu (1986).[7] However, we may point to such a floor in more applied and recent contributions to the literature. When the marginal cost is constant and set equal to zero, as in Häckner (1996) or Compte et al. (2002), for examples, the lowest possible profits are zero. Another example is Vasconcelos (2005), where there is a variable marginal cost and a finite demand, so that profits can be negative but limitedly so. Our more general specification also captures these cases.

[6] In two related papers, Yasuda (2009) and Beviá, Corchón, and Yasuda (2011) introduce a similar specification in order to study how financial constraints affect collusion equilibrium payoffs and firms' behavior in repeated games. Yasuda (2009) shows in particular that, with a single-period punishment stick-and-carrot mechanism adapted from Abreu (1986), collusion in which Cournot duopolists equally divide a monopoly profit in each period may not be sustainable. Beviá, Corchón, and Yasuda (2011) also specify that profits must be greater than or equal to an exogenously given value, which is non-positive. They characterize the allocations which can be sustained as an equilibrium of a dynamic oligopoly model when no firm can be forced to bankruptcy by any other firm satisfying the financial constraint. In both papers, a firm is assumed to go bankrupt if its profits are driven below the financial threshold. This can be interpreted as a very severe form of punishment, as a binding financial constraint is assumed to result in zero *continuation* profits. Our analysis is thus complementary, since in the present paper a binding limited liability constraint does not imply bankruptcy. It only sets a limit to the severity of punishments, which may be possibly associated to positive profits, as in the case of a profit target imposed by financial markets pressure.

[7] In contrast, in the present model, the most severe punishment $\underline{\pi}$ can be arbitrarily close to the Nash payoff $\pi(a_{NE})$. In the final section on policy implications, the regulator can choose to adjust the profit floor *above* the Nash payoff.

We now introduce a few additional assumptions that are needed to produce formal results:

$(A4)$ If $a'_{-i} \preceq_i (\succ_i) a_{-i}$ then $\pi_i(a_i,a'_{-i}) \leq (>) \pi_i(a_i,a_{-i})$, all $a_i \preceq_i a_m$.

This assumption specifies the extension of the order relation to vector of actions. [8]

Another specification of the model relates to deviation profits. A firm can earn positive benefits by playing its best reply to all other firms' action, only if the latter action is not too severe. Formally:

$(A5)$ There exists $\tilde{a}_P \preceq_i a_{NE}$ such that $\pi_i^d(a) \leq (>) 0$ if and only if $a \preceq_i (\succ_i) \tilde{a}_P$.

When all firms in $N\backslash\{i\}$ play $a \succ_i \tilde{a}_P$, the latter assumption implies that firm i's *gross* deviation profits are strictly higher than the level of fixed costs, that is f. A consequence of $(A5)$ is that $\pi(a_{NE}) \geq 0$.

Although the analysis focuses on situations with limited punishments, the latter may be very severe. A reference action that measures this severity is $\hat{a}_P$, which is such that the minmax profit is obtained by stopping production. We assume that:

$(A6)$ There exists $\hat{a}_P \preceq_i \tilde{a}_P$ such that $\pi_i^d(a) = (>) - f$ if and only if $a \preceq_i (\succ_i) \hat{a}_P$.

In terms of output quantity, let $q_i^d(a)$ denote firm i's best-reply to a, as chosen by all other firms. Assumption $(A6)$ specifies that $q_i^d(a) = 0$ if $a \preceq_i \hat{a}_P$, and $q_i^d(a) > 0$ otherwise. In words, any action a, as chosen by all firms in $N\backslash\{i\}$, that is strictly more severe than $\hat{a}_P$, drives firm i's profit-maximizing output to zero. In particular, if $\hat{a}_P \succeq_i \underline{a}_P$, then the most severe symmetric punishment action, when played by all firms in $N\backslash\{i\}$, is sufficiently penalizing as to incentivize firm i to stop producing, and thereby to incur losses equal to the magnitude of fixed costs, its minmax value. Note that $\pi(a) > \underline{\pi}$ if $\hat{a}_P \succeq_i a \succ_i \underline{a}_P$, although $q_i^d(a) = q_i^d(\underline{a}_P) = 0$, with firm i's best-reply profits $\pi_i^d(a) = \pi_i^d(\underline{a}_P) = -f \leq 0$. To gain familiarity with the notation, observe that when firms' strategic variable is price, and $c = f = \underline{\pi} = 0$, as commonly assumed for simplicity in many existing models, we have $\tilde{a}_P = \hat{a}_P = \underline{a}_P = 0$, a particular case.

[8] In the Bertrand (resp. Cournot) specification, firm i's profits are often non-decreasing (resp. non-increasing) with other firms' symmetric price (resp. quantity), so that if $p'_{-i} \leq p_{-i}$ (or $q'_{-i} \geq q_{-i}$) then $p'_{-i} \preceq_i p_{-i}$ (and $q'_{-i} \preceq_i q_{-i}$). This, however, does not hold in all cases. For example, in a simple price-setting oligopoly model with perfect substitutes and constant marginal costs, if $p_i < p_{NE}$ we have $p'_{-i} \succ_i p_{-i}$ for all $p'_{-i} < p_{-i}$.

When no constraint on the severity of a is introduced, as in most contributions to the literature, profits $\pi(a)$ are unbounded from below. In that case, since best-reply profits $\pi_i^d(a)$ do have a lower bound (a firm may always stop selling; see $(A6)$), we have $\pi_i^d(a) - \pi(a)$ unbounded from above. Recalling that $\pi_i^d(a_{NE}) - \pi(a_{NE}) = 0$, we know there exists at least one $\breve{a} \preceq_i a_{NE}$ verifying $\pi_i^d(\breve{a}) - \pi(\breve{a}) = \pi_i^d(a_m) - \pi_m > 0$. Finally we specify uniqueness, for simplicity:

$(A7)$ There exists a unique $\breve{a} \prec_i a_m$ such that $\pi_i^d(\breve{a}) - \pi(\breve{a}) = \pi_i^d(a_m) - \pi_m$.

Clearly $\breve{a} \prec_i a_{NE}$ (since $\breve{a} \preceq_i a_{NE}$ by definition and $\pi_i^d(a_{NE}) - \pi(a_{NE}) = 0 < \pi_i^d(a_m) - \pi_m$). Note that $(A7)$ is very mild. It captures in particular all usual situations in which the incentive to deviate $\pi_i^d(a) - \pi(a)$ increases with the severity of actions $a \preceq_i a_{NE}$, and also with the level of collusion $a \succ_i a_{NE}$.[9]

In what follows we investigate the role of the parameter $\underline{a}_P$, that is the most severe punishment action, on the implementation of collusion. This is done by first considering situations in which the duration of punishments is limited to a single period.

3 The Benchmark

In this section, as a benchmark, we restrict the duration of the punishment phase to a single period. For each player to have no incentive to deviate, a deviation must be followed by a punishment that leads the discounted flow of profits to be less than the actualized stream of collusive equilibrium profits. Moreover, for the punishment to be a credible threat, one should verify that firms do implement the punishment action. This occurs if individual gains to deviate from the punishment phase are smaller than the loss incurred by prolonging the punishment.[10] Formally, the profile $\{a_m, a_P\}$, with $a_P \preceq a_m$ (this is for all i, so we can drop the subscript for the order relation), must satisfy two incentive constraints, we refer to hereafter as $IC0$ and $IC1$, that is

$$\pi_i^d(a_m) - \pi_m \leq \delta\left[\pi_m - \pi(a_P)\right], \qquad (IC0)$$

$$\pi_i^d(a_P) - \pi(a_P) \leq \delta\left[\pi_m - \pi(a_P)\right], \qquad (IC1)$$

[9] For an illustration with quantity-setting firms see Fig. 2 in Abreu (1986). In the present paper Fig. A-1 (in the appendix) is made very intuitive when a is interpreted as a price.

[10] In a trigger penal code à la Friedman (1971), a deviation implies that firms stop colluding and revert to the one-shot stage game Nash equilibrium forever. The punishment action is then self-enforcing. A stick-and-carrot setup authorizes a more severe (and also shorter) punishment phase that may lead firms to earn negative profits for some time. It is not self-enforcing unless $(IC1)$ holds.

where $\pi(a)$ denotes a firm's stage profit when all competitors choose the same action a, and $\pi_i^d(a)$ is firm i's profit from a one-shot best deviation from the action a selected by all rivals in $N\backslash\{i\}$. The first condition says that the profits associated with a deviation from the collusive action must be smaller than what is lost due to the punishment phase. The second condition says that the benefits associated with a deviation from the punishment must be smaller than the loss incurred by prolonging the punishment by one more period.

Our objective is to delineate the largest space of parameters for which the two constraints are satisfied. The problem we investigate is thus to find a punishment a_P that minimizes δ under the two incentive constraints ($IC0$-$IC1$). The solution a_P^*, defined as the optimal punishment, yields δ^*, the minimum. Before introducing additional constraints, we characterize a_P^* and δ^* by presenting three intermediate results.

Lemma 1. *The optimal single-period punishment action a_P^* and the discount factor lower bound δ^* are such that ($IC0$) and ($IC1$) hold with equality.*

This first result establishes that, when $a_P = a_P^*$, and $\delta = \delta^*$, the two incentive constraints are exactly satisfied. Therefore we may compute a_P^* and δ^* by solving in (a_P, δ) the system ($IC0$-$IC1$) with equality signs.

To compare, recall that Abreu (1986)'s problem consists in identifying the pair of actions (a_P, a_C) that permits firms to maintain the most profitable collusive action a_C for a given discount factor δ. The two approaches are dual since the value δ^* we obtain as a solution, for a given a_m, is identical to the given value of δ that leads to the solution $a_C^* = a_m$ in Abreu's problem. In the latter, the solution a_C^* is bounded from above by the stage-game joint-profit maximizing action. When δ is high enough for this boundary value to be implemented as a collusive equilibrium, the constraint not to deviate from collusion is slack. This explains why Lemma 1 differs slightly from Abreu's Theorem 15, in which the analogue of ($IC0$) holds with a weak inequality *only* (while the analogue to ($IC1$) holds with an equality sign, as in the present case).

Note however that the single-period punishment action that implements the collusive action needs not be a_P^*. This is because a_P^* is defined as the punishment action that satisfies ($IC0$-$IC1$) for the *lowest* possible value of δ, that is *exactly* δ^*. When $\delta > \delta^*$, the collusive action is implementable with a "non-optimal punishment" a_P about a_P^*.

We now introduce two additional constraints. The first one is a participation constraint.[11] It specifies that each firm, when it actualizes the future stream of profits earned from the period of

[11] Lambson (1987) refers to it as an individual rationality constraint.

punishment onward, must be incentivized to continue playing the game even if it earned negative profits for a while. Formally, it must be the case that $\pi(a_P) + \sum_{k=1}^{\infty} \delta^k \pi_m \geq 0$. A simple reorganization of terms, toward a more intuitive expression, leads to

$$(1-\delta)\left[\pi_m - \pi(a_P)\right] \leq \pi_m. \tag{PC}$$

In words, the participation constraint is satisfied when the profit a firm forgoes in the punishment period, that is the difference $\pi_m - \pi(a_P)$, is not greater than the discounted stream of collusive profits earned in all following periods, that is $\pi_m/(1-\delta)$.

Note that $(IC1)$, which we may rewrite as $(1-\delta)\left[\pi_m - \pi(a_P)\right] \leq \pi_m - \pi_i^d(a_P)$, can be easily compared to (PC). Recalling from $(A5)$ that $\pi_i^d(a_P) \leq (>) 0$ if and only if $a_P \preceq (\succ) \tilde{a}_P$, observe that $(IC1)$ is (weakly) stronger than (PC) if and only if $a_P \succeq \tilde{a}_P$. It follows that, when $a_P^* \prec \tilde{a}_P$, (PC) is violated, hence δ^* is not attainable.

In this case, toward a solution to the participation-constrained problem we define a particular punishment action, denoted by $\overline{a}_P$, that satisfies exactly both $(IC0)$ and (PC). In formal terms, $\pi(\overline{a}_P) = \pi_m - \pi_i^d(a_m)$.[12] For notational clarity, let $\overline{\pi} \equiv \pi(\overline{a}_P)$. Note that $\overline{a}_P \prec a_{NE}$ because $\pi(\overline{a}_P) < 0$.

The next constraint is central to the analysis. It imposes a limit to the severity of the punishments all firms may inflict on each other in a single period. Formally, a_P must satisfy

$$\pi(a_P) \geq \underline{\pi}. \tag{LLC}$$

This constraint can be rooted in structural conditions (e.g., demand is finite at any price, including zero), financial considerations (e.g., a profitability target), or in institutional features (e.g., a regulation). In what follows we refer to this weak inequality as the limited liability constraint. It does not appear in Abreu (1986)'s seminal paper, where the inverse demand is *strictly* monotonic, and the constant marginal cost is always positive, so that losses can be made as negative as needed by charging sufficiently close to zero. In the majority of more recent models which capitalize on Abreu's results, and specify a stick-and-carrot mechanism with a single punishment period, a limited liability constraint is implicit (e.g., the quantity demanded is finite for all prices, including zero), although to the best of our knowledge its implications were not investigated in the literature.

[12] The implicit definition of $\overline{a}_P$ is obtained by rewriting $(IC1)$ as $\delta \geq \left[\pi_i^d(a_m) - \pi_m\right] / \left[\pi_m - \pi(a_P)\right]$, and (PC) as $\delta \geq -\pi(a_P) / \left[\pi_m - \pi(a_P)\right]$. Then observe that the denominators are equal. If $a_P^* \prec \tilde{a}_P$, we know that $\overline{a}_P$ exists. This is because $\pi_i^d(a_m) - \pi_m = \pi_i^d(a_P^*) - \pi(a_P^*)$ from Lemma 1, and $\pi_i^d(a_P^*) < 0$ from $(A5)$, hence $\pi(a_P^*) < \pi_m - \pi_i^d(a_m) < 0$. Recalling that $\pi(a_{NE}) \geq 0$, by the intermediate value theorem we have $a_P^* \prec \overline{a}_P \prec a_{NE}$ such that $\pi(\overline{a}_P) = \pi_m - \pi_i^d(a_m)$.

Note from ($IC\,0$) that the first incentive constraint is satisfied if and only if $\left[\pi_i^d(a_m) - \pi_m\right]/\delta \leq \pi_m - \pi(a_P)$, and from ($LLC$) that the limited liability constraint can be rewritten $\pi_m - \pi(a_P) \leq \pi_m - \underline{\pi}$. It follows that, for a given collusive "target" a_m to be implementable, we must have $\left[\pi_i^d(a_m) - \pi_m\right]/\delta \leq \pi_m - \underline{\pi}$ for some $\delta \in (0,1]$. The latter condition obviously does not hold if $\lim_{\delta \to 1}\left[\pi_i^d(a_m) - \pi_m\right]/\delta > \pi_m - \underline{\pi}$, or equivalently if $\underline{\pi} > \pi_m - \left(\pi_i^d(a_m) - \pi_m\right)$. Accordingly, the limited liability constraint can be so strong as to make collusion impossible. Because we assume that $\underline{\pi} < \pi(a_{NE})$, a feasibility condition for a_m to be implementable in this single-period punishment context is $\pi_m - \pi(a_{NE}) \geq \pi_i^d(a_m) - \pi_m$. In words, the one-shot profit of collusion must be greater than the gain to deviating from it.

The order relation on the set of punishment actions a_P, as defined in the previous section, implies that (LLC) can be rewritten as $a_P \succeq \underline{a}_P$. This does not mean that punishments cannot result in very low profits when (LLC) is satisfied. Indeed recall from ($A5$) that the "lower" bound $\underline{a}_P$, when played by all firms in $N\backslash\{i\}$, can be sufficiently severe as to make firm i stop producing as a best-reply.

We may now write the δ-minimization problem in a_P as follows:

$$\begin{aligned} &\min_{a_P \in A} \; \delta \\ &\quad s.t. \quad IC\,0; IC1; PC; LLC \end{aligned} \tag{1}$$

The lowest δ for which the collusive action a_m is implementable finds different expressions depending on the comparison of the structurally defined punishment actions a_P^*, $\overline{a}_P$, and $\underline{a}_P$.

Proposition 1. *The collusive action $a_m \preceq a_m^*$ is implementable with a single-period punishment if and only if $\delta \geq \delta_1^*$, with*

$$\delta_1^* = \begin{cases} \delta^* \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \pi(a_P^*)} & \text{if } a_P^* \succeq \underline{a}_P, \overline{a}_P \quad \text{(regime 1);} \\ \overline{\delta} \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \overline{\pi}} & \text{if } \overline{a}_P \succeq \underline{a}_P, a_P^* \quad \text{(regime 2);} \\ \underline{\delta} \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \underline{\pi}} & \text{if } \underline{a}_P \succeq a_P^*, \overline{a}_P \quad \text{(regime 3);} \end{cases} \tag{2}$$

with $\delta^* < 1$ and $\overline{\delta} < 1$ for all parameter values, and $\underline{\delta} < (=)\, 1$ if and only if $\underline{\pi} < (=)\, \pi_m - \left(\pi_i^d(a_m) - \pi_m\right)$.

The three regimes identified in Proposition 1 reflect which constraints are at play in the δ-minimization problem (1). In regime 1, the two incentive constraints are stronger than (PC) and (LLC). The optimal punishment is a_P^*, and the minimized discount factor is $\delta_1^* = \delta^*$ (here the subscript "1" refers to the single-period punishment case). In regime 2, ($IC\,0$) and (PC) bite,

the optimal punishment is $\overline{a}_P$, and a_m can be implemented only if $\delta \geq \delta_1^* = \overline{\delta}$; while in regime 3, ($IC0$) and ($LLC$) are binding, the optimal punishment is $\underline{a}_P$, and a_m can be implemented only if $\delta \geq \delta_1^* = \underline{\delta}$. Note that ($IC0$) is active in all regimes. In fact a firm's incentive to deviate from the collusive action remains the same in the three regimes.

Another important point is that the comparison between regimes 1 and 2 differs in kind from the comparison between regime 3 and either regime 1 or 2. More precisely, whether a solution is of the regime-1 or regime-2 type depends on whether (PC) is stronger than ($IC1$) or not. Their ranking is rooted in the firms' payoff functions. Whether regime 3 arises or not can *also* depend on the strategy set, which can be limited "from below" for all sorts of institutional or financial reasons that do not relate to cost or demand conditions.

Remark 1. *If $a_P^* \succeq \underline{a}_P, \overline{a}_P$, so that regime 1 applies, $\delta^* \geq \overline{\delta}, \underline{\delta}$.*

This remark emphasizes a subtle aspect of Proposition 1. Obviously, when either regime 2 or 3 applies, so that either (PC) or (LLC) binds, respectively, we have $\delta^* \leq \overline{\delta}, \underline{\delta}$. Indeed the δ-minimization problem (1) is more constrained than when only the incentive constraints ($IC0$) and ($IC1$) are considered. However, when regime 1 applies, it does *not* mean that (PC) and (LLC) are set aside. It only means that ($IC0$) and ($IC1$) are stronger than both (PC) and (LLC). Hence the relevant threshold δ^* *cannot* be lower than $\overline{\delta}$ and $\underline{\delta}$. More generally, in the single-period punishment benchmark problem, at most two constraints bind, that determine the threshold for δ. This threshold can only be higher than the other two expressions in (2).

A final observation is that, while δ^* and $\overline{\delta}$ are both lower than 1, the limited liability constraint can be so strong as to result in $\underline{\delta} > 1$, in which case the collusive action a_m is *not* implementable with a single-period scheme, for any δ. Recalling that our objective is to identify the largest space of parameters for which a given collusive action is implementable, it remains to investigate the possibility to lengthen the duration of the punishment phase. The intuition is that, by shifting to a multi-period punishment scheme, firms can penalize more severely a deviation than in the single-period framework. This can soften the lower bound condition on the discount factor, and thus facilitate collusion.[13] However, we demonstrate in the next section that this occurs only is very specific circumstances, we fully characterize.

[13] Several periods of punishment have been considered only in a few theoretical contributions with more specific assumptions than in the present model. Lambson (1987) considers price-setting sellers of a homogenous good, a constant average cost, with capacity constraints. Häckner (1996) constructs a repeated price-setting duopoly model, with spatial differentiation, and a constant average cost normalized to zero. In Lambertini and Sasaki (2002), again there are two firms and a constant marginal average cost, but with another specification of the horizontal differentiation assumption, together with a non-negative constraint on quantities, but not on prices.

4 The Main Results

In this section we introduce the possibility for firms to choose a punishment action over several periods. The objective is to investigate the impact of the extended length of punishment on firms' ability to implement collusion, when the severity of punishment is limited in each period.

To do that, consider a stick-and-carrot penal code in which, if any deviation from a_m by any firm is detected, all firms switch to a l-period punishment phase (the stick) during which they play $a_{P,k}$, with $k = 1, \ldots, l$. Punishment actions may vary from one period to another. A deviation from the punishment action may occur in any period of punishment. If this occurs, the punishment phase restarts for l more periods, after which all firms revert to the initial collusive action a_m forever (the carrot).

Formally, the two incentive constraints ($IC0$) and ($IC1$) are now extended to

$$\pi_i^d(a_m) + \sum_{k=1}^{l} \delta^k \pi(a_{P,k}) + \sum_{k=l+1}^{\infty} \delta^k \pi_m \leq \sum_{k=0}^{\infty} \delta^k \pi_m, \tag{3}$$

and

$$\pi_i^d(a_{P,s}) + \sum_{k=1}^{l} \delta^k \pi(a_{P,k}) + \sum_{k=l+1}^{\infty} \delta^k \pi_m \leq \sum_{k=s}^{l} \delta^{k-s} \pi(a_{P,k}) + \sum_{k=l+1}^{\infty} \delta^{k-s} \pi_m, \tag{4}$$

respectively, for any period s in which a firm deviates from the penal code, with $1 \leq s \leq l$, all i.

Given a_m, the vector $\mathbf{a}_P \equiv (a_{P,1}, \ldots, a_{P,k}, \ldots, a_{P,l})$ sustains collusion if and only if (3) and (4) are satisfied. There are $1 + l$ incentive constraints in all: the single constraint in (3) says that the gain earned by deviating from the collusive action must be smaller than what is lost over the l periods of punishment; the other l constraints in (4) say that the gain to deviate from the punishment phase, in any period s, with $1 \leq s \leq l$, must be smaller than the loss incurred by re-initiating the punishment phase.

To simplify the presentation of incentive constraints and clarify their interpretation, we now introduce a value function. If a firm does *not* deviate from the punishment path, the continuation profits it earns from period $s+1$ onward is

$$V_s(\mathbf{a}_P, \delta) = \sum_{k=s+1}^{l} \delta^{k-s-1} \pi(a_{P,k}) + \sum_{k=l+1}^{\infty} \delta^{k-s-1} \pi_m. \tag{5}$$

Here $s = 0$ indicates that the l-period flow of punishment profits is not truncated from below, whereas $s = l$ means that exactly all punishment profits are removed, so that only collusive profits

are considered from period $l+1$ onward. Note from (5) that $a_{P,l+1} = a_m$ implies $V_s(\mathbf{a}_P, \delta) \leq V_l(\mathbf{a}_P, \delta) = \pi_m/(1-\delta)$, all s. This also implies that $V_l(\mathbf{a}_P, \delta) = V_0(\mathbf{a}_m, \delta)$.

Then the multi-period incentive constraints in (3) and (4) are

$$\pi_i^d(a_m) - \pi_m \leq \delta\left[V_0(\mathbf{a}_m, \delta) - V_0(\mathbf{a}_P, \delta)\right], \qquad (MIC0)$$

and

$$\pi_i^d(a_{P,1}) - \pi(a_{P,1}) \leq \delta\left[V_1(\mathbf{a}_P, \delta) - V_0(\mathbf{a}_P, \delta)\right], \qquad (MIC1)$$

$$\ldots \qquad (...)$$

$$\pi_i^d(a_{P,s}) - \pi(a_{P,s}) \leq \delta\left[V_s(\mathbf{a}_P, \delta) - V_0(\mathbf{a}_P, \delta)\right], \qquad (MICs)$$

$$\ldots \qquad (...)$$

$$\pi_i^d(a_{P,l}) - \pi(a_{P,l}) \leq \delta\left[V_l(\mathbf{a}_P, \delta) - V_0(\mathbf{a}_P, \delta)\right], \qquad (MICl)$$

respectively, with $1 \leq s \leq l$. Note that $\pi(a_{P,s}) \leq \pi_i^d(a_{P,s})$ requires that $V_0(\mathbf{a}_P, \delta) \leq V_s(\mathbf{a}_P, \delta)$, all s, a feasibility condition of the punishment scheme.

In $(MIC\,0)$ we compare a firm's payoff when it colludes by choosing a_m, that is $\pi_m + \delta V_0(\mathbf{a}_m, \delta)$, with the payoff it earns by deviating, that is $\pi_i^d(a_m) + \delta V_0(\mathbf{a}_P, \delta)$. It is individually rational to stick to the collusive action if this first constraint is satisfied. The next incentive constraints, one for each period of punishment, compare a firm's payoff when it implements a punishment action, with the payoff it earns by deviating. More precisely, in $(MIC\,1)$ we compare the firm's payoff when it plays $a_{P,1}$, that is $\pi(a_{P,1}) + \delta V_1(\mathbf{a}_P, \delta)$, with the payoffs it earns by deviating, that is $\pi_i^d(a_{P,1}) + \delta V_0(\mathbf{a}_P, \delta)$. The next row describes the same comparison for the next period of punishment, and so on, down to $(MIC\,l)$. A firm will not deviate from the l-period punishment path if all constraints of rank $s = 1, \ldots, l$ are satisfied.

A first technical claim is a multi-period counterpart to Lemma 1, as offered above in the single-period punishment case.

Lemma 2. *Given $a_{P,1}$, the lowest discount factor δ verifying $(MIC\,0)$ and $(MIC\,1)$ results from punishment actions $a_{P,k}$, with $k > 1$, such that these two multi-period incentive constraints bind.*

The multi-period participation constraint is $V_s(\mathbf{a}_P, \delta) \geq 0$, all $s = 0, 1, \ldots, l$. In words, the continuation profits, from the first period of punishment onward, must remain non-negative for a firm to implement the punishment $\mathbf{a}_P$. Interestingly this can also be rewritten as

$$(1-\delta)\left[V_0(\mathbf{a}_m, \delta) - V_s(\mathbf{a}_P, \delta)\right] \leq \pi_m, \qquad (MPC)$$

all $s = 0, 1, \ldots, l$, an intuitive generalization of the single-punishment period counterpart in (PC). This says that the sum of profits that each firm foregoes by implementing the remaining punishment $a_{s+1}, \ldots, a_l$, that is the difference $V_0(\mathbf{a}_m, \delta) - V_s(\mathbf{a}_P, \delta)$, cannot be more than the discounted stream of profits earned in all collusive periods that follow, $\pi_m / (1-\delta)$.[14]

Observe from $(MIC\,0)$ and (MPC) that the value differential $V_0(\mathbf{a}_m, \delta) - V_0(\mathbf{a}_P, \delta)$ is bounded from below by $\left[\pi_i^d(a_m) - \pi_m\right]/\delta$ and from above by $\pi_m / (1-\delta)$, respectively. This yields:

Lemma 3. *The lowest δ compatible with $(MIC\,0)$ and (MPC) is $\overline{\delta} \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_i^d(a_m)}$.*

Therefore there can be no l-period punishment $\mathbf{a}_P$ that implements a_m when the discount factor is strictly lower than $\overline{\delta}$. In other words, the lengthening of the punishment scheme cannot help relaxing the participation constraint.

Now the multi-period limited liability constraint is

$$\pi(a_{P,k}) \geq \underline{\pi}, \qquad (MLLC)$$

with $1 \leq k \leq l$, all $l \geq 2$. In words, the limited liability constraint $(MLLC)$ captures structural conditions imposing that, in any period k of the punishment phase, a firm's profit cannot be driven below $\underline{\pi}$, a parameter. Note that $(MLLC)$ implies that $a_{P,1} \succeq \underline{a}_P$, which we use to prove (in Appendix) the following technical result:

Lemma 4. *The lowest δ compatible with $(MIC\,0)$ and $(MLLC)$ is $\underline{\delta}' \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_i^d(a_m) - \pi_i^d(\underline{a}_P)}$.*

Given all constraints, the multi-period punishment problem is

$$\begin{array}{cc} \min\limits_{(a_{P,1}, \ldots, a_{P,l}) \in A^l} & \delta \\ s.t. & (MIC\,0 - MIC\,l); MPC; MLLC \end{array} \qquad (7)$$

For any given l, the optimal multi-period punishment is the solution in $\mathbf{a}_P = (a_{P,1}, \ldots, a_{P,l})$ to (7). It yields the lowest possible value of the discount factor, we denote by δ_l^*, that authorizes firms to implement a_m, under all constraints. In what follows we examine successively the role of the $1 + l$ multi-period incentive constraints $(MIC\,0\text{-}MIC\,l)$, the participation constraint (MPC), and the limited liability constraint $(MLLC)$.

[14] The latter interpretation of (MPC) is even more intuitive when one sees that $V_0(\mathbf{a}_M, \delta) - V_0(\mathbf{a}_P, \delta) = \sum_{k=1}^{l} \delta^{k-1} (\pi(a_M) - \pi(a_{P,k}))$, so that $l = 1$ leads to (PC), the participation constraint in the single-period punishment setup.

We now establish that, in the absence of participation and limited liability constraints, or when they are slack, the possibility to punish over several periods does *not* result in an optimal punishment path that differs from the single-period punishment case, our benchmark.

Proposition 2. *In the multi-period punishment scheme, if* $a_P^* \succeq \overline{a}_P, \underline{a}_P$ *the collusive action* $a_m \preceq a_m^*$ *is implementable if and only if* $\delta \geq \delta^*$, *and* $\mathbf{a}_P^* \equiv (a_P^*, a_m, \ldots, a_m)$ *is optimal.*

Obviously it is always possible to replicate the single-period punishment scheme by playing $a_{P,1} = a_P$ in the first period, followed in all $l-1$ subsequent periods by the same collusive action, that is $a_{P,k} = a_m$, all $k = 2, \ldots, l$. Proposition 2 establishes that, when (MPC) and ($MLLC$) are slack, by doing so with $a_P = a_P^*$ one obtains the lowest possible value of δ for which the collusive action a_m is implementable. The threshold value of the discount factor we obtain in this l-period punishment scheme is the *same* as in the single-punishment case, namely δ^*.

Remark 2. *If* $a_P^* \succeq \overline{a}_P, \underline{a}_P$ *there is a unique punishment path* $\mathbf{a}_P^*$ *that permits firms to implement* a_m *for* $\delta = \delta^*$.

In other words, as long as the participation and limited liability constraints are not binding, there is one best way to solve (7). In a supergame with discounting, late punishments have less impact. Firms must charge a low price or supply a large quantity as early as possible, that is in the first punishment period, in order to minimize the discount factor at which a_m is implementable.

Next, we establish that, when the multi-period participation constraint binds, again the possibility to punish over several periods does *not* enlarge the space of parameters for which the collusive action is implementable.

Proposition 3. *In the multi-period punishment scheme, if* $\overline{a}_P \succeq \underline{a}_P, a_P^*$, *the collusive action* $a_m \preceq a_m^*$ *is implementable if and only if* $\delta \geq \overline{\delta}$, *and* $\overline{\mathbf{a}}_P \equiv (\overline{a}_P, a_m, \ldots, a_m)$ *is optimal.*

When (MPC) binds, by playing $\overline{a}_P$ in the first punishment period (as in the single-period scheme), followed by the same collusive action afterwards (i.e., $\overline{a}_{P,k} = a_m$, all $k = 2, \ldots, l$), one obtains the lowest possible value of δ for which a_m is implementable. This discount threshold is the same as in the single-punishment case when (PC) binds, that is $\overline{\delta}$. The intuition for this result is straightforward. Indeed the participation constraint $V_0(\mathbf{a}_P, \delta) \geq 0$ determines the maximum *total* punishment a firm can incur (as opposed to a per-period punishment). In fact this constraint is identical in the single- and multi-period schemes, since the definition of the maximum total punishment does not depend on the number of periods. When the participation constraint binds with only one punishment period, it cannot be relaxed by extending the number of periods.

Remark 3. *If* $\overline{a}_P \succ a_P^*$ *there is a continuum of punishments that permit firms to implement* a_m *for* $\delta = \overline{\delta}$.

This says that, when (MPC) binds, the punishment $\overline{\mathbf{a}}_P \equiv (\overline{a}_P, a_m, \ldots, a_m)$ is only one way, among others, of implementing a_m when the discount factor is the lowest possible, at $\overline{\delta}$. Firms may opt for a softer first-period action if they choose to lengthen the punishment phase to one or several subsequent periods, before reverting to a_m. While the possibility to punish over several periods does *not* permit firms to reduce the discount factor threshold for which the collusive action is implementable, the space of punishment strategies that allow them to reach a given threshold is strictly larger than in the single-period punishment case.

We now turn to the case of a binding limited liability constraint. We will see that it differs qualitatively from the previous cases, in that additional punishment periods result in a strictly lower discount threshold than with a single-period scheme.

The next proposition describes the optimal punishment, and characterizes the associated discount threshold, when $(MLLC)$ binds.

Proposition 4. *In the multi-period punishment scheme, if* $\underline{a}_P \succeq a_P^*, \overline{a}_P$ *collusion at* $a_m \preceq a_m^*$ *is implementable if and only if* $\delta \geq \underline{\delta}_M \equiv \sup\{\overline{\delta}, \underline{\delta}'\}$, *with* $\underline{\mathbf{a}}_P \equiv (\underline{a}_P, a_{P,2}, \ldots, a_{P,l})$ *of finite length* l.

Remark 4. *If* $(MLLC)$ *is strictly binding, that is if* $\underline{a}_P \succ a_P^*, \overline{a}_P$, *there exits a continuum of optimal punishments* $(\underline{a}_P, a_2, \ldots, a_l)$ *of finite length* $l \geq 2$, *such that* a_m *is implementable for* $\delta = \underline{\delta}_M$.

In other words, when the limited liability constraint binds, so that the single period punishment action $a_{P,1}$ cannot be more severe than $\underline{a}_P$, the multi-period optimal punishment profile does not necessarily look like the usual front-loading scheme (where firms are punished as much as immediately possible before returning to the collusive path as soon as possible). In fact the optimal profile $(\underline{a}_P, a_2, \ldots, a_l)$ can display much more complicated patterns.

Example 1. *Two price-setting firms sell a homogeneous good in a market with a linear demand. Sales from firm* i *are given by*

$$q_i(\mathbf{p}) = \begin{cases} q(p_i) & \text{if } p_i < p_j \\ \frac{1}{2}q(p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases},$$

where $q(p_i) = \sup\{0, \alpha - p_i\}$ *for* $\alpha > 0$ *and* $p_i \geq 0$, *with* $i, j = 1, 2$, $i \neq j$. *The unit cost of production is a constant* $c > 0$, *the fixed cost is* $f > 0$, *and there is a price-floor regulation which*

prohibits below-marginal-cost pricing, i.e. $\underline{p}_P = p_{NE} = c$*, so that the limited liability constraint is* $\pi(p_{P,k}) \geq \underline{\pi} = -f$*, with* $1 \leq k \leq l$*, all* $l \geq 2$*. The punishment profile* $\underline{\mathbf{a}}_P$ *possibly can take the form of a price asymmetric cycle, where fast price increases from c to the (perfect) collusive price* $p_m^* = \frac{\alpha+c}{2}$ *are followed by several smaller decreases down to the price floor, or the neighborhood of it (see Figure 1).*

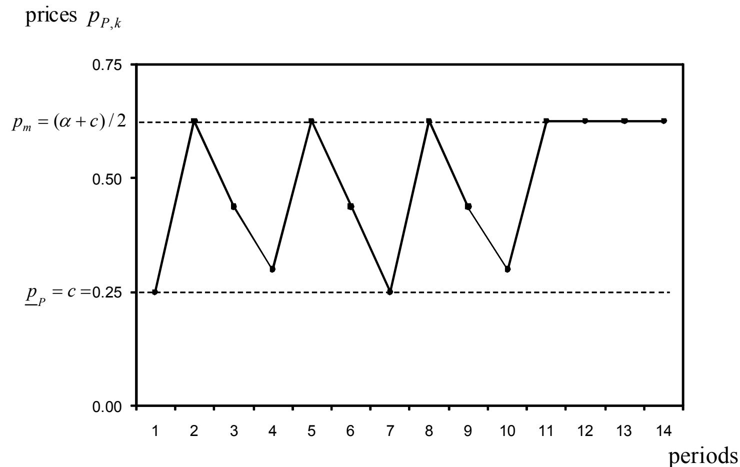

Figure 1: The punishment profile $\underline{\mathbf{p}}_P \equiv (\underline{p}_P, p_{P,2}, \ldots, p_{P,l})$ in Example 1 can take the form of asymmetric price cycles (here with $\alpha = 1$, $c = 1/4$, $\underline{p}_P = p_{NE} = c = 1/4$, implying that $\underline{\delta}_M \equiv \sup\{\underline{\delta}', \overline{\delta}\} = 1/2$). In this ten-period punishment phase, fast price increases from $\underline{p}_P$ to $p_m^* = (\alpha + c)/2$ are followed by a two-period fall down to $\underline{p}_P = c$ (here with intermediate prices $a_{P,3} = a_{P,6} = a_{P,9} = (\underline{p}_P + p_m^*)/2$).

Example 1 echoes recent empirical investigations on dynamic pricing behavior in retail gasoline markets, where asymmetric retail price cycles are observed. They begin with a price jump, followed by a series of smaller price cuts, until the observed price reaches the competitive level (Eckert (2002), Eckert and West (2004); Noel (2006, 2007)). Then the cycle restarts, and so on. This resembles the Edgeworth cycles obtained as a (non-collusive) equilibrium in an alternating-move price-setting duopoly model by Maskin and Tirole (1988). In the present model, we see in Figure 1 that two-phase asymmetric cycles are also consistent with the implementation of a multi-period punishment when firms are constrained not to charge below the marginal cost.[15]

[15] The limited liability constraint in Example 1 echoes the regulations that constrain the formation of gasoline retail prices above the wholesale (rack) price in several U.S. states and Canadian provinces (Houde 2008, 2010). In our setup there can be no deviation from the collusive path in equilibrium. However, should uncertainty of some kind be introduced, out-of-equilibrium punishment profiles would be observed (for example, unobserved random

We may now state our main proposition. It synthesizes the previous results, and allows us to rank all the discount thresholds introduced above.

Proposition 5. *If $\underline{a}_P \succ a_P^*, \overline{a}_P$, and additional punishment periods are introduced, the lowest discount factor $\underline{\delta}_M$ that permits the implementation of $a_m \preceq a_m^*$ cannot be as low as δ^*, and can attain $\overline{\delta}$ only in particular circumstances. More formally, either $\overline{a}_P \preceq a_P^*$ so that $\delta^* < \underline{\delta}_M < \underline{\delta}$, or $\overline{a}_P \succ a_P^*$ and $\overline{\delta} \leq \underline{\delta}_M < \underline{\delta}$. In the latter case $\underline{\delta}_M = \overline{\delta}$ if and only if $\tilde{a}_P \succeq \underline{a}_P \succ \overline{a}_P \succ a_P^*$.*

In other words, when regime 3 applies in the single-period scheme, a delayed punishment with discounting offers only an *imperfect* substitute for more immediate severity.

To see that, suppose that, absent the (multi-period) limited liability constraint ($MLLC$), regime 1 applies. Then recall from Remark 2 that the only punishment profile allowing firms to implement collusion when $\delta = \delta^*$, a lower bound, is $\mathbf{a}_P^* \equiv (a_P^*, a_m, \ldots, a_m)$. When limited liability results in regime 3 to apply, we know that a_P^* is unattainable in the first punishment period. In that case a longer punishment phase permits firms to increase the total punishment, and thereby facilitates collusion in that it results in a discount threshold $\underline{\delta}_M$ which is lower than $\underline{\delta}$. However, with discounting, delayed punishments harm less. They do not allow $\underline{\delta}_M$ to attain the lower bound δ^*.

As an alternative, suppose now that, absent the limited liability constraint, regime 2 applies. In that case, recalling that $\overline{a}_P$ is implicitly defined by $\overline{\pi} = \pi_m - \pi_i^d(a_m)$, it is straightforward to observe from the comparison of the expressions of $\overline{\delta}$ and $\underline{\delta}'$, as displayed in Proposition 3 and Lemma 4 respectively, that the two thresholds coincide if and only if $\pi_i^d(\underline{a}_P) = 0$, or equivalently $\underline{a}_P = \tilde{a}_P$. When punishments cannot be very severe, in that $\underline{a}_P \succ \tilde{a}_P$, firms earn positive profits by deviating from the punishment "floor" (i.e., $\pi_i^d(\underline{a}_P) > 0$, see Assumption $A6$). In that case there is *no* finite number of punishment periods that allow firms to implement a_m for a discount level as low as $\overline{\delta}$. That is, $\underline{\delta}_M > \overline{\delta}$. Only when the most severe punishment is such that firms cannot break even by deviating, so that their minmax profit is non-positive (i.e., $\pi_i^d(\underline{a}_P) \leq 0$), they may implement a_m by lengthening the punishment phase for any discount level greater than or equal to $\overline{\delta}$, that is $\underline{\delta}_M = \overline{\delta}$.

By substituting $(\underline{a}_P, a_m, \ldots, a_m)$ for $\mathbf{a}_P$ in ($MIC\,1$), and reorganizing terms, we obtain that $\pi_i^d(\underline{a}_P) \leq \pi_m$ for all $\underline{a}_P \preceq a_m$. This leads to:

Remark 5. $\underline{\delta}_M \leq 1$.

shocks on demand may induce price wars to appear in equilibrium, as first investigated in Porter (1983) and Green and Porter (1984)).

In other words, the Folk theorem (Fudenberg and Maskin (1986)) is verified in the multi-period punishment setup (recall from Proposition 1 that, with a single period of punishment, in Regime 3 we have $\underline{\delta} > 1$ for $\underline{\pi}$ sufficiently high).

The next section illustrates the latter results and their interpretation in the usual context of a linear case.

5 A Linear Case

In this section, we introduce additional specifications on costs and demand in order to illustrate the importance of considering limited liability constraints in the familiar context of a linear oligopoly structure. We investigate the circumstances which allow firms to sustain perfect collusion (i.e., to maximize joint profits) when prices cannot be negative. Toward this aim, we assume that, over all periods, demand is derived from a utility function adapted from Häckner (2000), of the form

$$U(\mathbf{q}, I) = \sum_{i=1}^{n} q_i - \frac{1}{2}\left(\sum_{i=1}^{n} q_i^2 + 2\gamma \sum_{i \neq j} q_i q_j\right) + I, \tag{8}$$

which is quadratic in the consumption of q-products and linear in the consumption of the composite I-good (i.e., the numeraire).[16] The parameter $\gamma \in (0, 1)$ measures product substitutability as perceived by consumers. If $\gamma \to 0$, the demand for the different product varieties are independent and each firm has monopolistic market power, while if $\gamma \to 1$, the products are perfect substitutes. Consumers maximize utility subject to the budget constraint $\sum p_i q_i + I \leq m$, where m denotes income, p_i is the non-negative price of product i, and the price of the composite good I is normalized to one. By symmetry, we note $\sum_{j \neq i} q_j = (n-1)q_j$. On the cost side, in the example we set $f = 0$, for simplicity, and a constant marginal cost $c < 1$. We examine the Cournot version of the model. With quantity-setting firms, the relation q' *is more severe than* q is formally equivalent to $q' \geq q$.

From (8) firm i's inverse demand function in each period is

$$p_i(q_i, q_j) = \sup\{0, 1 - q_i - \gamma(n-1)q_j\}, \tag{9}$$

and the inverse demand for each other symmetric firm j in $N \backslash \{i\}$ is

$$p_j(q_i, q_j) = \sup\left\{0, 1 - \gamma q_i - (1 + \gamma(n-2))q_j\right\}, \tag{10}$$

[16] In Häckner (2000), quantities q_i are multiplied by a parameter a_i, that is a measure of the distinctive quality of each variety i. Here we exclude vertical product differentiation by assuming that $a_i = 1$, all $i \in N$.

all $q_i, q_j \geq 0$, $i \neq j$. It is straightforward to check that a firm's profit function is continuous and the associated maximization problem is convex.

Which of the three regimes we identified in Proposition 1 applies depends on the status of the participation and limited liability constraints. This in turn depends on the number of firms n, the degree of product differentiation γ, and the marginal (and unit) cost c. The connection of the latter cost parameter to the limited liability constraint, is very intuitive in this example.

With a linear demand, the quantity demanded is finite at all prices. The limited liability constraint here does not artificially set boundaries to firms' strategies, as it only formalizes that prices cannot be negative:

$$\pi(a_{P,k}) \geq \underline{\pi} \equiv \pi\left(\underline{q}_P\right),$$

where the most severe punishment $\underline{q}_P$ is obtained when the price charged by all firms is equal to zero. This may result in exactly zero profits if the marginal cost is equal to zero as well, or to losses if the price-cost margin is negative, all other things (i.e., the demand to each firm) remaining equal. Whether the endogenous q_P^* or $\overline{q}_P$, as defined above (by simply substituting q for a) is less or more severe than $\underline{q}_P$ can thus be seen to depend only on the comparison of c with a threshold level, we denote by $\underline{c}$, which is a function of n and γ.

In the specific algebraic context of this example, we check that (PC) binds if and only if $q_P \geq \tilde{q}_P$, where $\tilde{q}_P = (1-c)/[\gamma(n-1)]$ is computed by solving $\pi^d(q) = 0$ (see Assumption $(A5)$). Note that, in the absence of fixed costs, we have $\tilde{q}_P = \hat{q}_P$ (see Assumption $(A6)$). Observe that, because of the absence of fixed costs, deviation profits here cannot be negative (a firm may stop producing to earn zero benefit). Moreover (LLC) binds if and only if $q_P \geq \underline{q}_P$, where $\underline{q}_P = 1/[1+\gamma(n-1)]$ is obtained by solving $p_i(q,q) = 0$. This is because, in the absence of regulatory intervention, the lower bound to punishment profits results from the non-negativity constraint in prices (it binds when quantities are sufficiently large, because demand is finite).

In the appendix, we compute the expression of the frontier $\tilde{c}$, a function of n and γ, which delineates the parameter space in which the quantity q_m^* (perfect collusion) can be implemented in the benchmark set-up with a single-period punishment scheme.[17] If $c < \tilde{c}$, collusion cannot be sustained, for any set of parameter values, with a single-period punishment scheme. However, we verify in Appendix 8.3.1 that collusion at q_m^* can *always* be implemented with a multi-period punishment scheme for some δ in $[\underline{\delta}_M, 1]$, which illustrates the Folk theorem in this lin-

[17]With a multi-period punishment scheme, the collusive quantity q_m^* is always implementable by mimicking a trigger mechanism (with $q_P = q_{NE}$, the Cournot equilibrium quantity, forever). In that case collusion is sustainable for all $\delta \geq \frac{\pi_i^d(q_m^*) - \pi_m^*}{\pi_i^d(q_m^*) - \pi(q_{NE})}$. The latter discount threshold is less than 1 for all $\pi_m^* > \pi(q_{NE})$.

ear setup. We also compute the three-part expression of a continuous frontier $\underline{c}$, with $\underline{c} = 0$ if $0 \leq \gamma \leq \hat{\gamma}$, $\underline{c} = \underline{c}' > 0$ if $\hat{\gamma} < \gamma \leq \check{\gamma}$, and $\underline{c} = \underline{c}'' > \underline{c}'$ otherwise, with $\hat{\gamma} \equiv 2/(n-1)$ and $\check{\gamma} \equiv 2\left(1+\sqrt{2}\right)/(n-1)$, all n. They lead to a partition of the parameter space (n, γ, c) into three subsets, one for each regime.

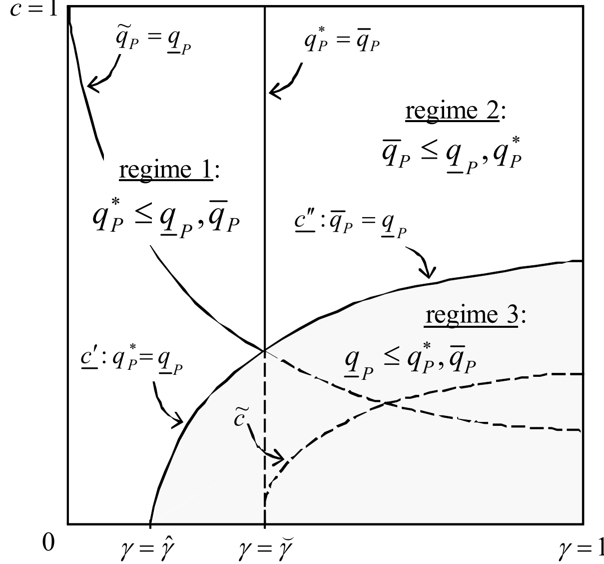

Figure 2: Collusion regimes in plane (c, γ) for $n \geq 6$. The limited liability constraint binds in the grey area (regime 3). In the benchmark single-period set-up, the collusive quantity is not implementable below the frontier $\tilde{c}$.

Proposition 6. *The parameter space* $(c,\ n,\ \gamma)$ *is partitioned in three subsets where either Regime 1, 2, or 3, as defined in* (2)*, applies.*

The partition (which is extensively described and commented in Appendix 8.3.3) is such that, if the constant unit cost is sufficiently high (formally, $c \geq \underline{c}$), either regime 1, where neither (PC) nor (LLC) binds, or regime 2, where (PC) binds, applies. The former case may hold for all $n \geq 2$, while the second cannot arise if $n < 6$. Regime 3 is ruled out only if $n = 2$. Otherwise, when goods are sufficiently substitutable ($\gamma \geq \hat{\gamma}$), and for all numbers of firms, a sufficiently large reduction in c will always result in a shift to regime 3, where (LLC) binds. Fig. 2 precisely illustrates this point.[18] Note that the non-implementability frontier $\tilde{c}$ is monotone increasing in γ and crosses the plane (γ, c) below $\underline{c}''$. It is non-negative if and only if $\gamma \geq \check{\gamma}$. Because $\inf\{\check{\gamma}, 1\} = 1$ if and

[18]In this figure, $\gamma < (=)\check{\gamma}$ is equivalent to $q_P^* < (=)\tilde{q}_P$ (see Appendix 8.3.2). Hence it is also equivalent to $q_P^* < (=)\overline{q}_P$, from Lemma 4.

only if $n < 6$, the collusive q_m^* cannot be implemented in the single-period scheme for all $n \geq 6$ if the products are sufficiently substitutable or the marginal cost is sufficiently low.[19]

Another interesting aspect of Proposition 6 is that the limited liability constraint can be ignored for all values of c and γ if there are exactly two or three firms (see Regime 1-(i)). In that case, the results obtained in the literature on the implementation of collusion with a duopoly, homogenous goods, and a cost set to zero, are robust. This does not apply when $n > 3$.

It is also of interest to compare Proposition 6 with Abreu (1986), where there is no limited liability constraint. In that paper, the model is a Cournot oligopoly with a strictly positive constant unit cost, homogenous goods, and a quantity demanded that tends to infinity when the price approaches zero. Then the collusive q_m^* can be implemented with a one-period punishment penal code, for all numbers of firms, provided that the discount factor δ is above a threshold δ^*. If there are at most three firms, Proposition 6 extends Abreu's result to our specific example for any (c, γ). This is remarkable since our demand specification is *not* a special case of Abreu's class of demand functions. However, with more than three firms, the values of c and/or γ must be higher than a threshold for a single-period punishment scheme to implement collusion at $\delta = \delta^*$ or $\delta = \underline{\delta}$.[20]

The role of costs, given n and γ, can be illustrated by comparing q_P^* and $\overline{q}_P$ with $\underline{q}_P$ for any c defined on $[0, 1]$. The punishment quantities are represented for all $n \geq 6$ again, with highly substitutable products in Fig. 3(a), where $\check{\gamma} < \gamma$, and for more differentiated products in Fig. 3(b), where $\gamma < \hat{\gamma}$. In both cases regime 3 applies when the constant cost parameter is low, that is $c \leq \underline{c}$. For higher levels of c we have regime 2 in (a), and regime 1 in (b). Note that the cost threshold $\underline{c}$ is monotone increasing in n and γ (see (26-28) in Appendix 8.3.2).

As for the structural boundary level $\underline{q}_P$, it depends only on the number of competitors and demand parameters. It is monotone decreasing when either n or γ increases, but constant in c. The optimal punishment quantities q_P^* and $\overline{q}_P$ are linear in the cost parameter and monotone decreasing when it rises closer to 1.

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[19] In the Appendix 8.3.1 we show that the feasibility condition in Assumption ($A8$) is always satisfied. It would not be the case if a sufficiently high floor on p_i or low capacity constraint on q_i were added to the specifications. Then q_m^* could not be implemented for any admissible δ even in the multi-period set-up.

[20] This result contrasts even more sharply with trigger penal code models, in which one can easily check that the sustainability of collusion is not directly connected to the level of marginal costs (at least in the linear cost setup).

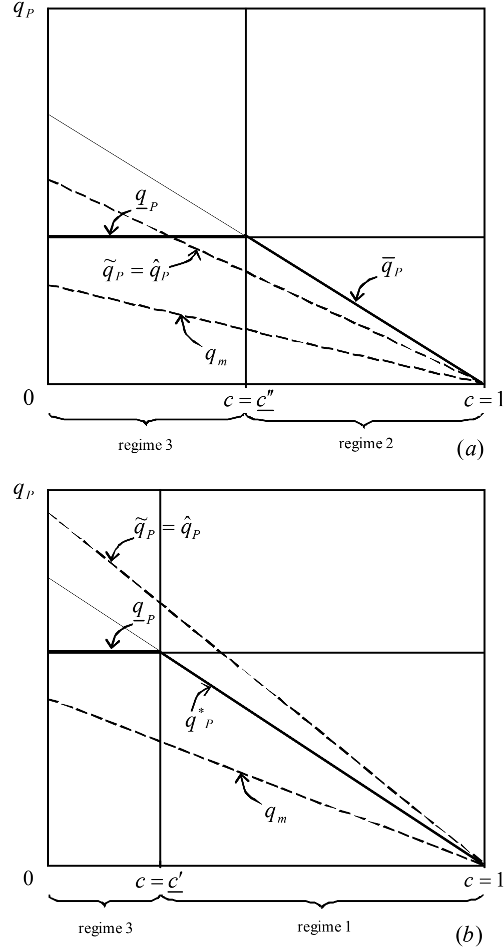

Figure 3: Thick lines represent optimal punishment quantities (all c, and $n \geq 6$). In (a) products are highly substitutable ($\breve{\gamma} < \gamma$). Regime 3 applies for $c \leq \underline{c}''$, and regime 2 applies otherwise. In (b) products are more differentiated ($\gamma < \breve{\gamma}$). Regime 3 applies for $c \leq \underline{c}'$, and regime 1 applies otherwise.

We can also characterize the effect of a change in the marginal cost c, the differentiation parameter γ, or the number of firms n, on the discount factor thresholds δ^*, $\underline{\delta}$, $\overline{\delta}$, and $\underline{\delta}_M$. To summarize:

Proposition 7. *High marginal costs facilitate collusion in that the limited liability constraint plays no role only if $c \geq \underline{c}$, where $\underline{c}$ is monotone increasing in n,γ. Moreover:* (i) *δ^* and $\overline{\delta}$ are monotone increasing in n and γ, and are independent of c*; (ii) *$\underline{\delta}$ and $\underline{\delta}_M$ are monotone increasing in n and γ, and monotone decreasing in c.*

This proposition establishes that an increase in product differentiation, and a reduction in the number of firms, facilitate collusion in two ways. Given δ, it enlarges the range of cost parameters for which optimal collusion can be implemented. Given c, more differentiation and less firms both lower the discount factor thresholds associated to the three different regimes.

6 Connections to the Literature

In this section, we discuss the robustness of theoretical results, as received from a selection of related papers, to the introduction of a limited liability constraint.

One stream of the theoretical literature on collusion has followed Friedman (1971) by considering trigger strategies (commonly referred to as "grim" strategies), which call for reversion to the one-shot stage game Nash equilibrium forever when a deviation from the collusive rule is detected in a previous period. When firms cannot observe their competitors' output levels, unobserved random shocks in demand can induce price wars to appear in equilibrium (see Porter (1983), Green and Porter (1984)). When all parameters and individual strategies are observable, models with various specific functional forms indicate that, for any number of firms, tacitly collusive agreements are more easily sustained with quantity-setting firms than with price-setting oligopolists, and with highly differentiated products (Deneckere (1983, 1984), Majerus (1988), Chang (1991), Ross (1992), Häckner (1994)).

A weakness common to all models of collusion with trigger strategies is that they rule out the possibility of modulating the level of punishments. More precisely, by assuming that when a deviation is detected firms revert to the Nash equilibrium of the one-shot stage game forever, they arbitrarily put an upper bound on the severity of punishments. In this particular context where the strategy set is *de facto* truncated, the limited liability constraint plays no role. Indeed, by assumption (LLC) cannot bind whenever $\underline{a}_P \prec a_{NE}$. Provided the strategy set is not arbitrarily truncated, the limited liability constraint does impact firms' ability to sustain collusion.

A series of papers that formalize a stick-and-carrot penal code, in the spirit of Abreu (1986, 1988), investigate the impact of product differentiation and industry concentration on the sustainability of collusive agreements. An example is Wernerfelt (1989), who finds that more product differentiation renders collusion less sustainable when the number of quantity-setting oligopolists is relatively large.[21] In a repeated Bertrand (i.e., price-setting) model with two firms, spatial hori-

[21] Although of interest, this ambiguous result is derived from demand assumptions (adapted from Deneckere, 1983) which are not standard (on this see Osterdal, 2003, pp. 54-55).

zontal differentiation, and a constant marginal cost set equal to zero, Häckner (1996) demonstrates that there exists an optimal symmetric stick-and-carrot punishment scheme, and establishes instead that differentiation facilitates collusive agreements. It is also demonstrated that, when the punishment price is constrained to be non-negative, a prolonged price war is an optimal collusive strategy. However in this setup there cannot be below cost pricing, a restriction that does not fit most real-world circumstances. Our paper extends the analysis to positive marginal costs, and reveals they *do* impact firms' ability to sustain collusion.

With two firms and constant marginal costs again, but with another specification of the horizontal differentiation assumption, Lambertini and Sasaki (2002) find a qualitatively similar relationship between product substitutability and collusion sustainability. This is obtained in a setup where quantities are constrained to be non-negative although prices may fall below zero. The example in the previous section extends this result to a linear setup with n firms, when the limited liability constraint imposes prices to be non-negative. Lambertini and Sasaki also find that, for all degrees of product differentiation, perfect collusion is less easily sustainable in Bertrand than in Cournot. The intuition is that incentives to deviate in the Bertrand case are higher than in the Cournot setup, because a deviating firm can capture the whole demand in a price-setting oligopoly. This does not apply in a quantity-setting oligopoly, because a unilateral expansion in some firm's output cannot eliminate its rivals' demand. When the limited liability constraint is taken into account, in principle the ability to sustain collusion can only be further limited. Although the severity of punishment does not depend on the nature of competition, we show in section 5 that the limited liability constraint binds more often in Bertrand than in Cournot.

Another strand of literature investigates the impact of cost asymmetries on collusion. Compte, Jenny, and Rey (2002) capitalize on early characterizations by Lambson (1987, 1994) of optimal punishments – possibly over several periods – for a class of infinitely repeated games with price-setting sellers of a homogenous good. They examine the impact of the distribution of firm-specific capacity constraints on the ability to sustain collusion. When capacity constraints are weak, in that any subset of firms can serve the entire market, the Nash equilibrium of the stage game yields zero profit. When aggregate capacity is limited *vis-à-vis* market size, it is shown that asymmetric capacities make collusion more difficult to sustain. With no fixed cost and a constant marginal cost normalized to zero, firms earn zero profit when they are minmaxed. This holds also when the price is set to zero. Hence, the limited liability constraint associated to price non-negativity can never be binding. Our analysis reveals that another factor would be at play if the marginal cost were specified to be positive. In that case, the limited liability constraint would depend on

each firm i's capacity k_i, with the lowest profit equal to $-ck_i < 0$, and it could be binding.

In Vasconcelos (2005), quantity-setting firms have a different share of the industry capital, which determines their marginal costs. In a punishment period, the total industry output is divided in proportion to capital endowments. The analysis focuses on maximum punishments. They make a deviant firm earn its minmax payoff, that is zero (there are no fixed costs), from the first period of punishment onward. In the terms of our paper, this is equivalent to assuming that the firms' punishment quantities are such that the participation constraint binds. When this holds, an important result is that a one-period punishment penal code exists, where the collusive action leads to monopoly profits (perfect collusion), if the discount factor is higher than a threshold level that depends on the size of the largest firm. The introduction of our limited liability constraint – which is a natural extension since demand is finite so that punishments are structurally limited from below – would lead to a higher threshold for some parameter values. By choosing simple values for the cost and demand parameters, we find that the above-mentioned discount threshold remains unchanged only if the marginal cost parameter is sufficiently high. More specifically, by setting (say) $k_i = 1/n$ for each firm i's capital share (so that symmetry is restored) and $a = b = 1$ for the linear demand curve parameters in Vasconcelos' model, one obtains that $\overline{q}_P \leq \underline{q}_P$ if and only if $c \geq \sup\{0, \overline{c}(\delta)\}$, where $\overline{q}_P$ is the quantity such that both ($IC0$) and (PC), as defined in the present paper, are exactly satisfied, and $\underline{q}_P$ is the quantity that drives prices to zero.[22] For $c < \sup\{0, \overline{c}(\delta)\}$, the limited liability constraint binds, and a one-period simple penal code is suboptimal.

7 Policy Implications

The systematic analysis of the role of a limited liability constraint has direct policy implications. It points to a very simple regulatory mechanism which uses only *prima facie* observable information (i.e., current transaction prices) and makes firms abandon collusion by iteration. To see that, introduce another constraint which imposes that, in each period, a firm's strategy a cannot be more severe than $\underline{a}_R$ (the subscript "R" here means "regulation"). Formally:

$$\pi(a) \geq \pi_R, \qquad (R)$$

[22] With $k_i = 1/n$ and $a = b = 1$, the discount threshold of Proposition 2 in Vasconcelos (2005, p. 48) reduces to $3\,(n+2)\,n/\,(2n+1)^2$. With δ at the latter level, we obtain $\overline{c}(\delta) = 1/n$.

where $\pi_R \equiv \pi(\underline{a}_R)$. In practice, the policy instrument which corresponds to that constraint is a price floor.[23] The regulatory constraint (R) is formally identical to a limited liability constraint. It can be made stronger than $(MLLC)$ by specifying that $\underline{a}_P \preceq \underline{a}_R \preceq a^*_m$, which implies that $\underline{\pi} \leq \pi_R \leq \pi^*_m$, where $\pi^*_m \equiv \pi(a^*_m)$.

Next, by substituting $\underline{a}_R$ for $\underline{a}_P$ in the expression of $\underline{\delta}'$ (see Lemma 4), define $\underline{\delta}'(\underline{a}_R, a_m) \equiv \frac{\pi^d_i(a_m)-\pi_m}{\pi^d_i(a_m)-\pi^d_i(\underline{a}_R)}$. For any given a, the monotonicity of the best-reply function $\pi^d_i(.)$ implies that $\underline{\delta}'(\underline{a}_R, a_m) > (=)\underline{\delta}'(a, a_m)$ for all $\underline{a}_R \succ (=)a$, with $\lim_{\underline{a}_R \to a_m} \underline{\delta}'(\underline{a}_R, a_m) = +\infty$.[24] Hence:

Remark 6. *For any $a_m \preceq a^*_m$ and $\delta \geq \underline{\delta}'(\underline{a}_P, a_m)$ there exists $\underline{a}_R \succeq \underline{a}_P$ such that $\underline{\delta}'(\underline{a}_R, a_m) = \delta$.*

It follows from the latter remark that, in principle, any given a_m can be made non implementable by adjusting $\underline{a}_R$ continuously. Suppose that $\underline{a}_R$ and a_m are such that $\underline{\delta}'(\underline{a}_R, a_m) = \delta$. Substituting $\underline{a}'_R \prec \underline{a}_R$ for $\underline{a}_R$ in (R) relaxes the constraint, implying that $\underline{\delta}'(\underline{a}'_R, a_m) < \delta$, and collusion is facilitated. On the other hand, substituting $\underline{a}'_R \succ \underline{a}_R$ for $\underline{a}_R$ in (R), other things remaining equal, implies that $\underline{\delta}'(\underline{a}'_R, a_m) > \delta$, hence a_m is not implementable. In that case, firms face only two possibilities: (i) if there exists a'_m (in the neighborhood of a_m) verifying $\underline{\delta}'(\underline{a}'_R, a'_m) \leq \delta$, then a'_m is implementable; (ii) otherwise collusion cannot be sustained.

However, in practice only the transaction prices, as actually charged by the industry, can be observed at some non prohibitive cost by the regulator. Whether firms collude or not cannot be inferred from that information, because the Nash equilibrium strategy of the stage-game, a_{NE}, is *a priori* unknown.

A simple regulatory mechanism can be designed that circumvents this inference problem. Suppose that the regulator sets limits to any reduction in transaction prices, starting from the observed level. In more formal terms, a_m is the current action played by all firms, with $a_{NE} \preceq a_m \preceq a^*_m$.[25] The regulator does not know whether the current a_m is a collusive action or not. Then, she specifies that in the next period, the observed a_1 must verify $a_1 \succeq \underline{a}_{R,1}$, where $\underline{a}_{R,1}$ is strictly less profitable than the current industry action (i.e., $\underline{a}_{R,1} \prec a_m$), but only limitedly so ($\underline{a}_{R,1}$ is a price slightly lower than the current transaction price in the Bertrand setup, or a quantity slightly greater each firm's current output in the Cournot specification). This is equivalent to introducing (R), with $\pi_R = \pi(a_m) - \epsilon$, where the magnitude of $\epsilon > 0$ is small (so that $a_{NE} \prec \underline{a}_{R,1}$).

[23] The regulatory constraint is rewritten as $p_i \geq \underline{p}_R$ in the Bertrand specification of the model, and as $p_i(\mathbf{q}) \geq \underline{p}_R$ in the Cournot scenario, all $i \in N$.

[24] In the appendix (Lemma A-2) we prove formally that $\pi^d_i(a) \geq \pi^d_i(a')$, all $a \succeq_i a'$.

[25] In the Bertrand setup all firms initially charge p_m, with $p_m \in [p_{NE}, p^*_m]$. In the Cournot scenario $p_m = p_i(\mathbf{q}_m)$, with each firm i selling $q_m \in [q^*_m, q_{NE}]$.

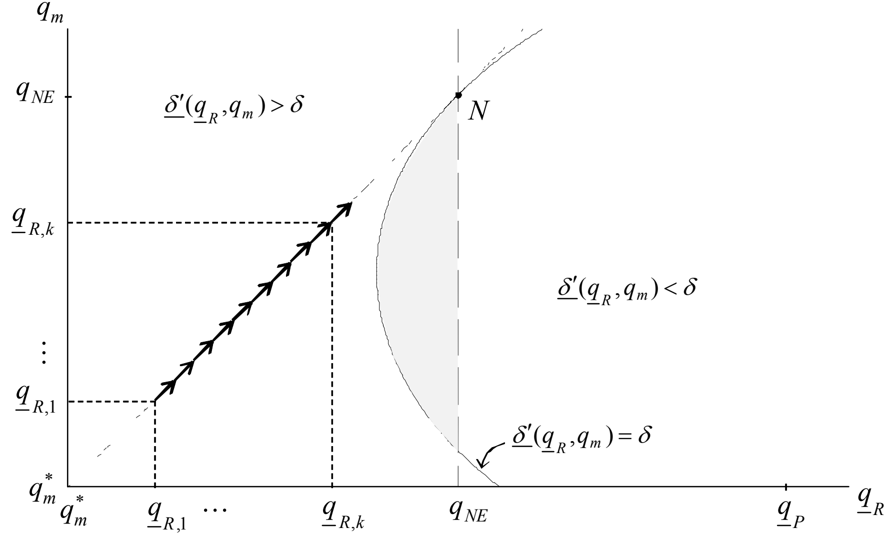

Figure 4: Cournot linear setup, with the utility function (8), five firms, a homogeneous good, and constant unit costs ($n = 5$, $\gamma = 1$, $c = 1/10$, $\delta = 3/5$). Initially, all firms implement the collusive quantity q_m^*. A price floor rules out large price reductions (i.e., $q \leq \underline{q}_{R,1}$, with $\underline{q}_{R,1}$ above q_m^*, but only limitedly so). A series of successive adjustments from $\underline{q}_{R,1}$ to $\underline{q}_{R,2}$, $\underline{q}_{R,3}$, ... drives the industry toward the stage-game Nash point N.

Note that a_m satisfies the regulatory constraint, which does *not* call explicitly for any change in the conduct of firms. At first glance, it merely rules out extreme variations in transaction prices, and thereby overtly protects the industry from sharp reductions in instantaneous profits. If firms do not collude, so that the current a_m results from a competitive behavior (i.e., $a_m = a_{NE}$), the constraint is clearly innocuous. Only in case of collusion, the constraint necessarily induces a change in firms' behavior.

In more technical terms, the firms' reaction to the introduction of the regulatory constraint depends on the status of a_m:

1) When firms do not collude initially, so that $a_m = a_{NE}$, the constraint (R) plays no role. Therefore, all firms keep playing the same stage-game equilibrium strategy.

2) When firms collude initially, so that $a_{NE} \prec a_m \preceq a_m^*$, for all $\underline{a}_{R,1}$ sufficiently close to a_m the regulatory constraint (R) binds in the punishment phase, so that $\underline{\delta}'\left(\underline{a}_{R,1}, a\right) > \delta$ for all $a \preceq a_m$, and collusion cannot be sustained. Moreover, $\underline{a}_{R,1} \succ a_{NE}$ implies that each individual stage-game best-reply strategy is also constrained by (R). As a consequence, all firms switch from a_m to $a_1 = \underline{a}_{R,1}$ (i.e., the constrained Nash equilibrium strategy), and

only a small drop in the observable transaction price occurs (from p_m to p_R in the price version of the model, and from $p_i\left(\underline{\mathbf{q}}_m\right)$ to $p_i\left(\underline{\mathbf{q}}_R\right)$ in the quantity setup). The process can be repeated, step by step, by a controlled *relaxation* of the regulatory constraint, with successive levels $\underline{a}_{R,1}, \underline{a}_{R,2}, \underline{a}_{R,3}, \ldots$ which are monotone "decreasing" in the order relation $\preceq$ toward a_{NE}.[26]

Example 2. *Consider the Cournot linear specifications introduced in Section 5, with* $n = 5$, $\gamma = 1$, $c = 1/10$, *and* $\delta = 3/5$. *The "latent" limited liability constraint* $\pi(q) \geq \underline{\pi}$, *or equivalently* $q \leq \underline{q}_P$, *can be made stronger by introducing a price floor, as formalized by* $q \leq \underline{q}_{R,1}$, *with* $\underline{q}_{R,1} < \underline{q}_P$. *With* $\underline{q}_{R,1}$ *strictly larger, but only limitedly so, than the collusive output* q_m, *with* $q_{NE} \leq q_m \leq q_m^*$, *the regulator can initiate a series of adjustments* ($\underline{q}_{R,1}$, $\underline{q}_{R,2}$, ...) *toward the stage-game Nash equilibrium* q_{NE} *(Fig. 4).*

Two conditions are necessary for this incremental process not to stop before the industry is driven to a_{NE}. First, the diagonal $\underline{a}_R = a_m$, as represented in Figure 4 with quantity as a choice variable, should not intersect the frontier $\underline{\delta}'(\underline{a}_R, a_m) = \delta$ for all $\underline{a}_R \succeq a_{NE}$ (i.e., for all $\underline{q}_R \leq q_{NE}$ in Figure 4). Otherwise the industry could implement $\underline{a}_{R,k}$ as a collusive action, while pretending that it actually competes. In that case the regulator could not check that the firms do not play a_{NE} when the collusive action remains equal to $\underline{a}_{R,k}$, with an observed track of adjustments remaining on the diagonal regulatory path. We establish that the regulator is actually immune from this moral hazard:

Lemma 5. *Consider any implementable collusive* a_m. *Then* $\underline{a}_R \succ a_{NE}$ *implies that* $a_m \succ \underline{a}_R$.

The second condition for the regulatory adjustment process to converge to a_{NE} is that the increments $\underline{a}_{R,k} - \underline{a}_{R,k-1}$ must be relatively small. Otherwise, when the process has reached a point on the diagonal $\underline{a}_R = a_m$ in the neighborhood of the frontier $\underline{\delta}'(\underline{a}_R, a_m) = \delta$, the industry could take advantage of a large increment to "leap" to a collusive point and stop the adjustment process. (In Fig. 4 this would be represented by a *horizontal* jump from the diagonal path $\underline{q}_R = q_m$ to a point in the shaded area between the critical frontier and the vertical line $\underline{q}_R = q_{NE}$.) In other words, after a finite number k of adjustments the process would permit firms to play an action $\underline{a}_{R,k}$ verifying $\underline{\delta}'\left(\underline{a}_{R,k}, \underline{a}_{R,k-1}\right) \leq \delta$. The regulatory constraint would be so weak as to permit punishments at $\underline{a}_{R,k}$ which are sufficiently severe to deter deviations from the collusive action

[26] Of course, anticipating the adjustment of the regulatory constraint $\underline{a}_R$ results in a lower incentive to deviate from collusion than actually specified by the constraint (3). Yet, as long as the price floor is sufficiently close to the current collusive price to make it unsustainable, the regulatory constraint (R) is sufficiently strong to drive the industry way from a_m.

$\underline{a}_{R,k-1}$. Again the regulator would not be able to detect collusion, since the observed transaction price, when all firms play $\underline{a}_{R,k-1}$, and stick to it, would be apparently compatible with a stop at the unobservable a_{NE}. This does not occur if the difference $\underline{a}_{R,k} - \underline{a}_{R,k-1}$ is infinitesimal.[27] To summarize:

Proposition 8. *Suppose that firms implement a_m, with $a_{NE} \prec a_m \preceq a_m^*$. By setting a price floor slightly below the observed transaction price, and by reducing the floor by iteration, the regulator makes the industry converge to the stage-game Nash equilibrium a_{NE}.*

Given that the diagonal $\underline{a}_R = a_m$ cannot intersect the frontier $\underline{\delta}'(\underline{a}_R, a_m) = \delta$ in any case (Lemma 5), and for sufficiently small increments $\underline{a}_{R,k} - \underline{a}_{R,k-1}$ (see previous footnote), any deviation by firms from the regulatory process would necessarily signal to the regulator that they are colluding. This will not occur in this framework if collusion is sufficiently penalized by monetary payments (i.e., standard cartel prosecution). If $\underline{a}_{R,k} - \underline{a}_{R,k-1}$ is not that small, firms are able to collude again (in Figure 4, by jumping to a collusive point in the shaded area), and the regulatory process stops after a finite number of iterations. However the termination collusive action can be made as close as desired to the one-shot Nash level by decreasing the size of increments.

8 Appendix

8.1 Single-Period Punishments

Proof of Lemma 1. Suppose that $a = a_P^*$, the optimal punishment in the interior of A (it is always possible to define A for this condition to hold), and $\delta = \delta^*$, the lowest possible discount factor for which a_m is implementable. There are three possible cases: either the two inequalities are slack, or only one, or none. Consider the first two cases in turn. (i) If none of the two constraints binds, observe that the two expressions on the RHS of the inequality sign are continuous in δ and monotonically decreasing when the discount parameter is decreasing, so that there exists $\delta' < \delta^*$ such that the system still holds true when $\delta = \delta'$, contradicting the claim that δ^* is a lower bound. (ii) If exactly one constraint binds for $\delta = \delta^*$, recall that profit functions $\pi_i^d(.)$ and $\pi(.)$ are continuous in firms' choices, therefore by changing slightly the punishment action from a_P^* to a'_P one can relax the binding constraint and still let the other inequality be verified. This leads the two constraints ($IC0$) and ($IC1$) to be slack, implying again that there exists $\delta' < \delta^*$ such that the system still holds true when $\delta = \delta'$. It follows from (i) and (ii) that both constraints

[27] Formally, the magnitude of the successive adjustments $\underline{a}_{R,k} - \underline{a}_{R,k-1}$ should be strictly less than the "distance" $|\alpha_k - \underline{a}_{R,k-1}|$, where α_k is defined by $\underline{\delta}'(\alpha_k, \underline{a}_{R,k}) = \delta$.

must be binding. ■

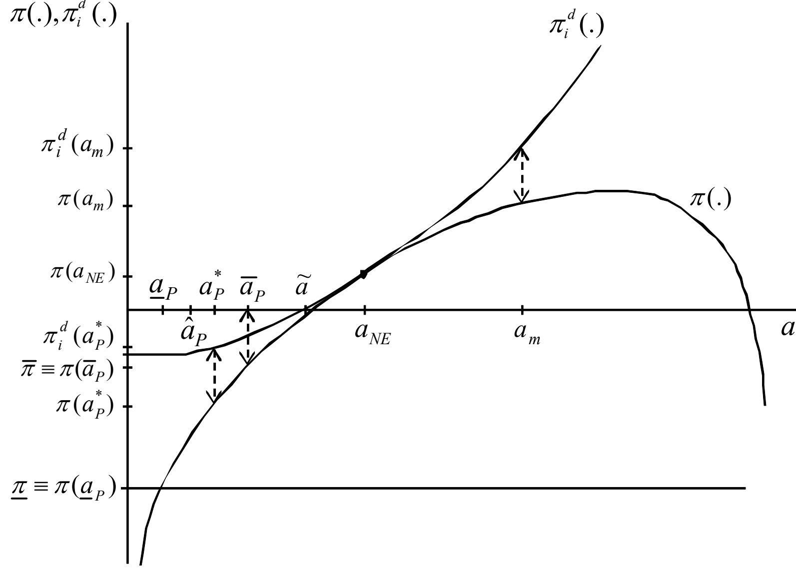

Figure A-1: The optimal punishment action a_P^* is such that $\pi_i^d(a_P^*) - \pi(a_P^*) = \pi_i^d(a_m) - \pi_m$, given a_m (here with $\underline{a}_P \prec a_P^* \prec \overline{a}_P \prec \tilde{a}$). As in Abreu (1986) the difference $\pi_i^d(a_P) - \pi(a_P)$ is unbounded from above if the limited liability constraint is removed.

Proof of Proposition 1. We first introduce three intermediate results (Lemmas A-1 to A-3).

Lemma A-1. *Given a_m, the optimal punishment action a_P^* is such that $\pi_i^d(a_P^*) - \pi(a_P^*) = \pi_i^d(a_m) - \pi_m$. Hence $a_P^* = \check{a}$ as defined in Assumption (A7).*

Proof. The constraints in (*IC0-IC1*) can be rewritten as $\delta \geq \delta'$ and $\delta \geq \delta''$, respectively, with $\delta' \equiv \left[\pi_i^d(a_m) - \pi_m\right] / \left[\pi_m - \pi(a_P)\right]$ and $\delta'' \equiv \left[\pi_i^d(a_P) - \pi(a_P)\right] / \left[\pi_m - \pi(a_P)\right]$. Lemma 1 implies that

$$\delta^* = \delta'\big|_{a_P = a_P^*} = \delta''\big|_{a_P = a_P^*}.$$

It is then sufficient to observe that the numerators of δ' and δ'' are identical to conclude that the numerators $\pi_i^d(a_m) - \pi_m$ and $\pi_i^d(a_P) - \pi(a_P)$ are also equal if $a_P = a_P^*$. ■

Lemma A-1 offers an implicit definition of a_P^* and says that, in the stage game, a firm's incentive to deviate from a_P^* is equal to the incentive to deviate from a_m (see Fig. A-1). Note that, because $a_P^* = \check{a}$ from Lemma A-1, where $\check{a}$ is as in Assumption (*A*7), by continuity of $\pi_i^d(.) - \pi(.)$, together with $\pi_i^d(a_{NE}) - \pi(a_{NE}) = 0$, the incentive to deviate from a_m is an upper bound to a firm's incentive to deviate, for any a that verifies $a_P^* \preceq a \preceq a_m$.

The next technical result establishes a monotonicity property.

Lemma A-2. $\pi_i^d(a) \geq \pi_i^d(a')$, *all* $a \succeq a'$.

Proof. Recall that $\pi_i^d(a) \equiv \pi_i\left(a_i^d(a_{-i}), a_{-i}\right)$, with $a_{-i} \equiv a$, and $a_i^d(a_{-i}) \equiv \arg\max_{a_i} \pi_i(a_i, a_{-i})$. From the definition of $a_i^d(a_{-i})$, we have $\pi_i\left(a_i^d(a_{-i}), a_{-i}\right) \geq \pi_i\left(a_i^d\left(a'_{-i}\right), a_{-i}\right)$, all a_{-i}, a'_{-i}. Next, $a \succeq_i a'$ here can be rewritten as $a_{-i} \succeq_i a'_{-i}$, implying that $\pi_i\left(a_i^d\left(a'_{-i}\right), a_{-i}\right) \geq \pi_i\left(a_i^d\left(a'_{-i}\right), a'_{-i}\right)$. This leads to $\pi_i^d(a) \geq \pi_i^d(a')$ by transitivity. ■

A useful technical result is:

Lemma A-3. $\overline{a}_P \succeq a_P^*$ *if and only if* $\tilde{a}_P \succeq a_P^*$.

Proof. **Sufficiency:** If $a_P^* \preceq \tilde{a}_P$ then $\pi_i^d(a_P^*) \leq 0$ by $(A5)$. Suppose that $\overline{a}_P \prec a_P^*$ (which implies that $\overline{a}_P \prec a_{NE}$ because $\tilde{a}_P \preceq a_{NE}$ from Assumption $(A5)$), and look for a contradiction. First recall that, absent the limited liability constraint, profits $\pi(a)$ are unbounded from below by assumption, while best-reply profits $\pi_i^d(a)$ have a lower bound (a firm may always stop selling from Assumption $(A6)$). Hence $\pi_i^d(a) - \pi(a)$ is unbounded from above (i.e., the difference is strictly larger than the constant $\pi_i^d(a_m) - \pi_m$ for a sufficiently severe a in the absence of limited liability constraint). Then suppose that $\pi_i^d(\overline{a}_P) - \overline{\pi} \leq \pi_i^d(a_m) - \pi_m$, by continuity of $\pi_i^d(.) - \pi(.)$ there would exist $a \prec \overline{a}_P \prec a_P^*$ such that $\pi_i^d(a) - \pi(a) = \pi_i^d(a_m) - \pi_m$, contradicting Lemma A-1 and Assumption $(A7)$. Hence $\pi_i^d(\overline{a}_P) - \overline{\pi} > \pi_i^d(a_m) - \pi_m$. Next, by Lemma A-2, $\overline{a}_P \prec a_P^*$ implies $\pi_i^d(\overline{a}_P) \leq \pi_i^d(a_P^*)$ hence $\pi_i^d(\overline{a}_P) \leq 0$. It follows that $\overline{\pi} < \pi_m - \pi_i^d(a_m) + \pi_i^d(\overline{a}_P) \leq \pi_m - \pi_i^d(a_m)$, which clearly contradicts the definition of $\overline{a}_P$. As a result $\tilde{a}_P \succeq a_P^*$ implies $\overline{a}_P \succeq a_P^*$. **Necessity:** If $\tilde{a}_P \prec a_P^*$, suppose that $\overline{a}_P \succeq a_P^*$ and look for a contradiction. By assumption $\overline{a}_P \preceq a_m$, and clearly $\overline{\pi} < 0$ implies $\overline{a}_P \prec a_m$. By Lemma A-1 and $(A7)$, $a_P^* \preceq \overline{a}_P \prec a_m$ implies that $\pi_i^d(\overline{a}_P) - \overline{\pi} \leq \pi_i^d(a_m) - \pi_m$. From the very definition of $\overline{a}_P$, it follows that $\pi_i^d(\overline{a}_P) \leq 0 = \pi_i^d(\tilde{a}_P)$. By Lemma A-2, this implies that $\overline{a}_P \preceq \tilde{a}_P$ and by transitivity through $\tilde{a}_P \prec a_P^*$, that $\overline{a}_P \prec a_P^*$, a contradiction. Hence $\overline{a}_P \succeq a_P^*$ implies $\tilde{a}_P \succeq a_P^*$. ■

The latter three technical results are useful to establish Proposition 1, as follows. There are three steps. First we solve a less constrained version of (1), in which (PC) and (LLC) are absent. Then we reintroduce each of the latter two constraints separately, one after another.

1) Consider the δ-minimization problem without constraints (PC) and (LLC). The two constraints $(IC0\text{-}IC1)$ can be rewritten together as

$$X(\delta) \leq \pi_m - \pi(a_P) \leq Y(\delta, a_P), \tag{11}$$

where $X(\delta) \equiv \left[\pi_i^d(a_m) - \pi_m\right]/\delta$ and $Y(\delta, a_P) \equiv \left[\pi_m - \pi_i^d(a_P)\right]/(1-\delta)$ denote the lower-bound and the upper-bound, respectively, of the profit differential $\pi_m - \pi(a_P)$. (They are represented

in Fig. A-2.) We know that (a_P^*, δ^*) solves $X(\delta) = Y(\delta, a_P)$ from Lemma 1.[28] Together with $\pi_i^d(a_m) - \pi_i^d(a_P^*) = \pi_m - \pi(a_P^*)$ from Lemma A-1, this leads to

$$\delta^* = \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \pi(a_P^*)}. \tag{12}$$

Then observe (i) from $(IC1)$ that $\pi_i^d(a_P^*) \leq \delta\pi_m + (1-\delta)\pi(a_P^*)$; and (ii) that $a_P^* \prec a_m$ implies $(1-\delta)(\pi(a_P^*) - \pi_m) < 0$, which can be rewritten as $\delta\pi_m + (1-\delta)\pi(a_P^*) < \pi_m$. Then (i) and (ii) together imply that $\pi_i^d(a_P^*) < \pi_m$, and consequently $\pi_i^d(a_P^*) - \pi(a_P^*) < \pi_m - \pi(a_P^*)$. As the difference on the LHS is equal to $\pi_i^d(a_m) - \pi_m$ from Lemma A-1, we obtain that $\pi_i^d(a_m) - \pi_m < \pi_m - \pi(a_P^*)$, which implies from (12) that $\delta^* < 1$.

2) Introduce (PC), in addition to $(IC0\text{-}IC1)$. For $a_P = a_P^*$, recall that the latter two constraints imply $X(\delta) \leq \pi_m - \pi(a_P^*) \leq Y(\delta, a_P^*)$, while the participation constraint can be rewritten $\pi_m - \pi(a_P^*) \leq \overline{Y}(\delta)$, with $\overline{Y}(\delta) \equiv \pi_m/(1-\delta)$.

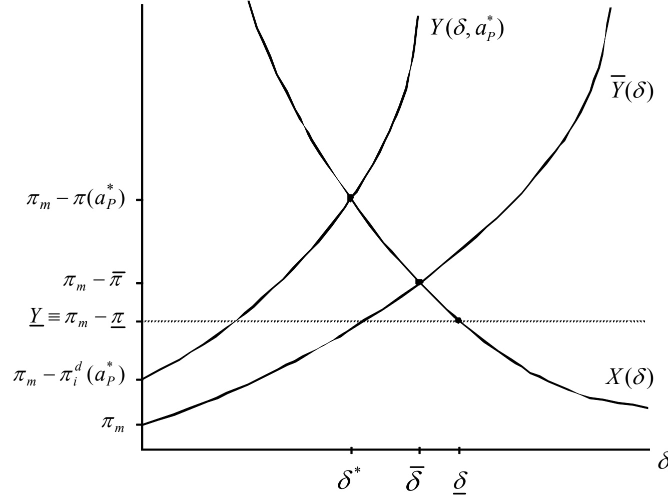

Figure A-2: The two ICs in $(IC0\text{-}IC1)$ can be rewritten $X(\delta) \leq \pi_m - \pi(a_P) \leq Y(\delta, a_P)$, with $X(\delta) \equiv \left[\pi_i^d(a_m) - \pi_m\right]/\delta$ and $Y(\delta, a_P) \equiv \left[\pi_m - \pi_i^d(a_P)\right]/(1-\delta)$. Similarly, PC can be rewritten $\pi_m - \pi(a_P^*) \leq \overline{Y}(\delta)$, with $\overline{Y}(\delta) \equiv \pi_m/(1-\delta)$, and LLC can be rewritten $\pi_m - \pi(a_P^*) \leq \underline{Y}$, where $\underline{Y} \equiv \pi_m - \underline{\pi}$. When PC and LLC are absent, the optimal punishment a_P^* and the threshold δ^* are such that $X(\delta^*, a_P^*) = Y(\delta^*)$. Here $a_P^* \prec \overline{a}_P \prec \underline{a}_P$, therefore LLC binds. The limited liability constrained optimal punishment is $\underline{a}_P$, and firms may implement a_m for all $\delta \geq \underline{\delta}$. The latter discount threshold is implicitly defined by $X(\underline{\delta}) = \underline{Y}$.

[28] Deviation profits $\pi_i^d(a_P)$ have a lower bound (a firm may always stop selling; see $(A6)$), all a_P. Therefore $\lim_{\delta \to 0} X(\delta) = +\infty > Y(0, a_P) = \pi_m - \pi_i^d(a_P)$, and $X(1) = \pi_i^d(a_m) - \pi_m < \lim_{\delta \to 1} Y(\delta, a_P) = +\infty$. Hence there always exists $\delta^*(a_P)$ in $[0,1)$ verifying $X(\delta^*(a_P)) = Y(\delta^*(a_P), a_P)$, all a_P.

There are two cases:

(i) If $\overline{a}_P \prec a_P^*$ then $\tilde{a}_P \prec a_P^*$, from Lemma A-3. Then we know from (PC) that $\overline{Y}(\delta) > Y(\delta, a_P^*)$ for all $\delta \in [0,1)$, and the participation constraint is slack for $a_P = a_P^*$ and $\delta = \delta^*$.

(ii) If $a_P^* \preceq \overline{a}_P$ then $a_P^* \preceq \tilde{a}_P$, from Lemma A-3. Then we know from (PC) that $\overline{Y}(\delta) \leq Y(\delta, a_P^*)$ for all $\delta \in [0,1)$. When the inequality sign is strict, (PC) is violated for $a_P = a_P^*$ and $\delta = \delta^*$. Next, toward a participation-constrained solution, substitute (PC) for $(IC1)$, or equivalently $\overline{Y}(\delta)$ for $Y(\delta, a_P)$in (11). (See Fig. A-2.) The negative slope of $X(\delta)$, the positive slope of $\overline{Y}(\delta)$, together with the continuity of $\pi(.)$, imply that the minimizer $\overline{a}_P$ and the minimum $\overline{\delta}$ verify $X\left(\overline{\delta}\right) = \pi_m - \overline{\pi} = \overline{Y}\left(\overline{\delta}\right)$.[29] This leads to

$$\overline{\delta} = \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \overline{\pi}}, \tag{13}$$

and then one checks that $\overline{Y}\left(\overline{\delta}\right) \leq Y\left(\overline{\delta}, \overline{a}_P\right)$. Recalling that $\overline{\pi} = \pi_m - \pi_i^d(a_m)$ by (implicit) definition $\overline{a}_P$, from (13) we have $\overline{\delta} < 1$ if and only if $\pi_i^d(a_m) - \pi_m < \pi_m - \left(\pi_m - \pi_i^d(a_m)\right)$, which is true for all $\pi_m > 0$.

(iii) Clearly if $\overline{a}_P \succ (=) \underline{a}_P$, then any $(\delta, \overline{a}_P)$, with $\delta \geq \overline{\delta}$, also verifies (LLC).

3) Introduce (LLC), in addition to $(IC0$-$IC1)$. Observe that the limited liability constraint can be rewritten $\pi_m - \pi(a_P) \leq \underline{Y}$, where $\underline{Y} \equiv \pi_m - \underline{\pi}$. There are two cases:

(i) If $\underline{a}_P \prec a_P^*$ we have $\underline{\pi} < \pi(a_P^*)$, hence (LLC) is slack for $a_P = a_P^*$, all δ.

(ii) If $a_P^* \preceq \underline{a}_P$, we know from (LLC) that $\underline{Y} \leq X(\delta^*) = \pi_m - \pi(a_P^*) = Y(\delta^*, a_P^*)$. When the inequality sign is strict (LLC) is violated for $a_P = a_P^*$ and $\delta = \delta^*$. Next, toward a limited liability constrained solution, one substitutes (LLC) for $(IC1)$, or equivalently $\underline{Y}$ for $Y(\delta, a_P)$ in (11). (See Fig. A-2.) Because $\underline{Y}$ is a constant, the slope of $X(\delta)$ is negative, and $\pi(.)$ is continuous, the minimizer $\underline{a}_P$ and the minimum $\underline{\delta}$ verify $X(\underline{\delta}) = \pi_m - \underline{\pi} = \underline{Y}$.[30] This leads to

$$\underline{\delta} = \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \underline{\pi}}. \tag{14}$$

Then $a_P^* \preceq \underline{a}_P \preceq a_m$ together with assumption $(A7)$ imply that $\underline{\delta} \geq \frac{\pi_i^d(\underline{a}_P) - \underline{\pi}}{\pi_m - \underline{\pi}}$, hence that $\underline{Y} \leq Y(\underline{\delta}, \underline{a}_P)$. It is obvious from (14) that $\underline{\delta} < (=) 1$ if and only if $\underline{\pi} < (=)\, \pi_m - \left(\pi_i^d(a_m) - \pi_m\right)$.

(iii) Clearly if $\underline{a}_P \succ (=) \overline{a}_P$, then any $(\delta, \underline{a}_P)$, with $\delta \geq \underline{\delta}$, also verifies (PC). ■

[29] Note that $\lim_{\delta \to 0} X(\delta) = +\infty > \overline{Y}(0) = \pi_m$ together with $\lim_{\delta \to 1} X(\delta) = \pi_i^d(a_m) - \pi_m < \lim_{\delta \to 1} \overline{Y}(\delta) = +\infty$ imply that there always exists $\overline{\delta}$ in $[0,1)$ verifying $X\left(\overline{\delta}\right) = \overline{Y}\left(\overline{\delta}\right)$.

[30] Since $X(\delta)$ is downward sloping, and $\lim_{\delta \to 0} X(\delta) = +\infty > \underline{X}$, there exists $\underline{\delta}$ in $[0,1)$ verifying $X(\underline{\delta}) = \underline{Y}$ if and only if $\lim_{\delta \to 1} X(\delta) < \underline{Y}$. This condition holds from Assumption $(A8)$. Otherwise a_m would not be implementable.

8.2 Multi-Period Punishments

We first prove Lemma 2, which is needed in the proof of Proposition 2, that follows.

Proof of Lemma 2. In (*MIC* 0), the expression on the RHS of the weak inequality sign simplifies to $\sum_{k=1}^{l} \delta^k [\pi_m - \pi(a_{P,k})]$. It is clearly monotone increasing when either $a_{P,k}$ decreases, all $k \geq 1$, or when δ increases, the LHS expression (which does not depend on punishment levels) remaining constant. In (*MIC* 1), the expression on the RHS of the weak inequality sign can be rewritten $\delta [(1-\delta) V_1 (\mathbf{a}_P, \delta) - \pi (a_{P,1})]$. It is monotone increasing when $a_{P,k}$ increases (since $\delta (1-\delta) > 0$), for all $k > 1$, the LHS expression (a function of $a_{P,1}$ only) remaining constant. Then for any given $a_{P,1}$, suppose that $a_{P,2}, \ldots, a_{P,l}$ are such that δ takes the lowest possible value for which (*MIC* 0-*MIC* 1) hold true. There are three possible cases: either the two inequalities are slack, or only one, or none. (i) If none of the two constraints binds, by continuity, one may obviously reduce δ by an arbitrarily small amount so that both constraints remain verified, contradicting the claim that there is no lower discount factor verifying (*MIC* 0) and (*MIC* 1). (ii) If exactly one of the two constraints binds, pick any $k > 1$ such that $a_{P,k} \prec a_m$. Then by continuity, one may reduce δ and adjust $a_{P,k}$ so that the RHS expression of the binding constraint remains constant, while the other constraint remains satisfied, contradicting again the initial supposition. Therefore it must be the case that, given $a_{P,1}$, (*MIC* 0-*MIC* 1) hold with an equality sign when $a_{P,2}, \ldots, a_{P,l}$ are such that δ is minimized. ■

Proof of Proposition 2. There are two steps: (1) We investigate a less constrained version of (7) by leaving aside the last $l-1$ multi-period incentive constraints together with (*MPC*) and (*MLLC*), to keep only (*MIC* 0) and (*MIC* 1). This is done by capitalizing on Lemma 2: we solve in (δ, V_1) the system (*MIC* 0-*MIC* 1) with equality signs, to obtain $(\delta^*(a_{P,1}), V_1(\mathbf{a}_P, \delta^*(a_{P,1})))$; then we identify the level of $a_{P,1}$that minimizes $\delta^*(a_{P,1})$ under the feasibility constraint that $V_1(\mathbf{a}_P, \delta^*(a_{P,1})) \leq V_l (\mathbf{a}_P, \delta^*(a_{P,1})) = \pi_m / (1-\delta^*(a_{P,1}))$. This leads to the minimizer $a^*_{P,1} = a^*_P$. (2) We show that $(\delta^*(a^*_P), V_1(\mathbf{a}_P, \delta^*(a^*_P)))$ satisfies all incentive constraints in (*MIC* 0-*MIC l*) as well as (*MPC*-*MLLC*).

(1) Consider the δ-minimization problem with the two incentive constraints (*MIC* 0) and (*MIC* 1) only. Observing that $V_l (\mathbf{a}_P, \delta) = V_0 (\mathbf{a}_m, \delta)$, the two constraints become

$$X(\delta) \leq V_0 (\mathbf{a}_m, \delta) - V_0 (\mathbf{a}_P, \delta) \leq Y(\delta, a_{P,1}), \tag{15}$$

where $X(\delta) \equiv \left[\pi_i^d (a_m) - \pi_m\right] / \delta$ and $Y(\delta, a_{P,1}) \equiv \left[\pi_m - \pi_i^d (a_{P,1})\right] / (1-\delta)$ denote the lower-bound and the upper-bound, respectively, of the value differential $V_0 (\mathbf{a}_m, \delta) - V_0 (\mathbf{a}_P, \delta) =$

$V_0(\mathbf{a}_m, \delta) - \pi(a_{P,1}) - \delta V_1(\mathbf{a}_P, \delta)$. Given $a_{P,1}$, from Lemma 2 we know that (15) must hold with an equality sign throughout for δ to be minimized. Solving $X(\delta) = Y(\delta, a_P)$ in $(\delta, V_1(\mathbf{a}_P, \delta))$, we find

$$\delta^*(a_{P,1}) = \frac{\pi_i^d(a_m) - \pi_m}{\pi_i^d(a_m) - \pi_i^d(a_{P,1})}, \tag{16}$$

and

$$V_1(\mathbf{a}_P, \delta^*(a_{P,1})) = \left[\pi_i^d(a_m) - \pi_i^d\left(a_{P,1}^*\right)\right]\left(\frac{\pi_i^d\left(a_{P,1}^*\right) - \pi\left(a_{P,1}^*\right)}{\pi_i^d(a_m) - \pi_m} + \frac{\pi_i^d\left(a_{P,1}^*\right)}{\pi_m - \pi_i^d\left(a_{P,1}^*\right)}\right). \tag{17}$$

Observe from the monotonicity of $\pi_i^d(a_{P,1})$ in $a_{P,1}$ (Lemma A-2) that $\delta^*(a_{P,1})$ is monotone non-decreasing in $a_{P,1}$. Therefore the lowest value of $\delta^*(a_{P,1})$ is obtained for the most severe first-period punishment $a_{P,1}$ compatible with the feasibility constraints of the problem. Note in particular from (5) that $a_{P,1}$ must be such that $V_s(\mathbf{a}_P, \delta) \leq V_t(\mathbf{a}_P, \delta) \leq V_l(\mathbf{a}_P, \delta) = \pi_m/(1-\delta)$, all $s \leq t \leq l$. Then $V_1(\mathbf{a}_P, \delta) \leq \pi_m/(1-\delta)$, together with (16) and (17), becomes

$$\left[\pi_m - \pi_i^d(a_{P,1})\right]\left(1 - \frac{\pi_i^d(a_{P,1}) - \pi(a_{P,1})}{\pi_i^d(a_m) - \pi_m}\right) \geq 0. \tag{18}$$

Clearly $\pi_m - \pi_i^d(a_{P,1})$ for all $a_{P,1} \preceq a_{NE}$ (since the monotonicity of $\pi_i^d(a_P)$ implies that $\pi_i^d(a_{P,1}) \leq \pi_i^d(a_{NE}) = \pi(a_{NE})$, while $\pi(a_{NE}) < \pi_m$ for all $a_{NE} \prec a_m$). It follows from (18) that the term between rounded brackets must be non-negative. This implies that

$$\pi_i^d(a_{P,1}) - \pi(a_{P,1}) \leq \pi_i^d(a_m) - \pi_m. \tag{19}$$

Recalling from Lemma A-1 that $\pi_i^d(a_P^*) - \pi(a_P^*) = \pi_i^d(a_m) - \pi_m$, from Assumption ($A7$) we obtain that $a_{P,1}$ cannot be strictly more severe than a_P^*.

(2) Substitute a_P^* for $a_{P,1}$ in $(16-17)$, and also $\pi_i^d(a_m) - \pi_m$ for $\pi_i^d(a_P^*) - \pi(a_P^*)$, again from Lemma A-1, to obtain

$$\delta^*(a_P^*) = \delta^* \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_i^d(a_m) - \pi_i^d\left(a_P^*\right)},$$

and

$$V_1^*(a_P^*, \delta^*(a_P^*)) = \frac{\pi_m}{1-\delta^*}.$$

It follows directly from the later equation that $V_1^*(a_P^*, \delta^*(a_P^*)) = V_l(\mathbf{a}_P, \delta^*(a_P^*))$, implying that $\pi\left(a_{P,k}^*\right) = \pi_m$, all $k > 1$. This says that $\mathbf{a}_P^* = (a_P^*, a_m, \ldots, a_m)$ when the only the two incentive constraints in (MIC 0) and (MIC 1 are considered. Next, observe from the definition of continuation profits in (5) that $a_{P,k}^* = a_m$, all $k > 1$, implies that $V_1(\mathbf{a}_P^*, \delta) = V_s(\mathbf{a}_P^*, \delta)$, all s. It

follows that the last $l-1$ multi-period incentive constraints are all identical to the first one, that is ($MIC\,0$), implying that all constraints in ($MIC\,0$-$MIC\,l$) are satisfied. Since $a_P^* \succeq \overline{a}_P, \underline{a}_P$ it is also plain that (MPC) and ($MLLC$) are satisfied. Therefore the solution to the less constrained problem is also a solution to (7), and the punishment $(a_P^*, a_m, \ldots, a_m)$ is optimal. ■

We now prove Lemma 3, which is needed in the proof of Proposition 3, that follows.

Proof of Lemma 3. The threshold $\delta = \left[\pi_i^d(a_m) - \pi_m\right]/\pi_i^d(a_m)$ follows directly from the comparison of (MIC 0 and (MPC) for $s = 0$. This threshold does not differ from $\overline{\delta}$, as introduced in Proposition 1, since the denominator $\pi_i^d\,(a_m) = \pi_m - \overline{\pi}$ from the implicit definition of $\overline{a}_P$. ■

Proof of Proposition 3. There are two steps: (1) In addition to ($MIC\,0$) and ($MIC\,1$), we introduce (MPC) in the less constrained version of (7), the last $l-1$ multi-period incentive constraints and ($MLLC$) being left aside. We show that (MPC) is stronger than ($IC1$) if $a_P^* \preceq \tilde{a}_P$. Then a_m is implementable with the l-period punishment $\overline{\mathbf{a}}_P \equiv (\overline{a}_P, a_m, \ldots, a_m)$ if $\delta = \overline{\delta}$, that is the lower bound to the interval of δ for which ($MIC\,0$) and (MPC) are compatible. (2) We obtain that $(\overline{\delta}, \overline{\mathbf{a}}_P)$ satisfies all other incentive constraints ($MIC\,2$-$MIC\,l$), in which case $\overline{\delta}$ is a solution of (7) and $\overline{\mathbf{a}}_P$ is optimal.

(1) Introduce the multi-period participation constraint (MPC) in addition to ($MIC\,0$-$MIC\,l$). For $\mathbf{a}_P = \mathbf{a}_P^* \equiv (a_P^*, a_m, \ldots, a_m)$ recall that the first two incentive constraints in ($MIC\,0$) and ($MIC\,1$) can be rewritten $X\,(\delta) \leq V_0\,(\mathbf{a}_m, \delta) - V_0\,(\mathbf{a}_P^*, \delta) \leq Y\,(\delta, \mathbf{a}_P^*)$, while ($MPC$) can be rewritten $V_0\,(\mathbf{a}_m, \delta) - V_0\,(\mathbf{a}_P^*, \delta) \leq \overline{Y}\,(\delta)$, with $\overline{Y}\,(\delta) \equiv \pi_m/\,(1-\delta)$. If $\overline{a}_P \succeq a_P^*$ we know from Lemma A-3 that $\tilde{a}_P \succeq a_P^*$, in which case $\pi_i^d\,(a_P^*) \leq 0$ from ($A5$). This implies that $\overline{Y}\,(\delta) \leq Y\,(\delta, a_P^*)$ for any $\delta \in [0,1)$. When the inequality sign is strict (MPC) is stronger than ($MIC\,1$), and thus is violated for $\mathbf{a}_P = \mathbf{a}_P^*$ and $\delta = \delta^*$. Next, toward a participation-constrained solution, substitute (MPC) for ($MIC\,1$). From Proposition 1, in the single-period punishment case we know that ($IC\,0$) and (PC) are satisfied if $a_P = \overline{a}_P$ and $\delta \geq \overline{\delta}$, implying that in the multi-period setup ($MIC\,0$) and (MPC) are satisfied as well if $\overline{\mathbf{a}}_P \equiv (\overline{a}_P, a_m, \ldots, a_m)$ and $\delta \geq \overline{\delta}$. This is sufficient to conclude that there is at least one punishment $\mathbf{a}_P$ for which a_m is implementable with $\delta \geq \overline{\delta}$. Then recall from Lemma 3 that $\overline{\delta}$ is the lowest value of δ compatible with ($MIC\,0$) and (MPC). This is sufficient to conclude that $\overline{\delta}$ is a solution to the δ-minimization problem under the constraints ($MIC\,0$), ($MIC\,1$), (MPC).

(2) Observe from the definition of continuation profits in (5) that $a_{P,k} = a_m$ for all $k > 1$ implies that $V_s\,(\mathbf{a}_P, \delta) = V_0\,(\mathbf{a}_m, \delta)$, all $s > 1$. It follows the last $l-1$ multi-period incentive constraints are all identical to ($MIC\,0$), implying that all constraints in ($MIC\,0$-$MIC\,l$) are satisfied. Clearly

if $\overline{a}_P \succ (=)\underline{a}_P$, then $\left(\overline{\delta}, \overline{\mathbf{a}}_P\right)$ also verifies $(MLLC)$. Therefore $\overline{\delta}$ is a solution to (7), and the punishment $(\overline{a}_P, a_m, \ldots, a_m)$ is optimal, all l. ■

Proof of Lemma 4. First, recall from Lemma 2 that, given $a_{P,1}$, the lowest discount factor δ verifying $(MIC\,0)$ and $(MIC\,1)$ results from punishment actions $a_{P,k}$, with $k > 1$, such that both $(MIC\,0)$ and $(MIC\,1)$ bind. This implies that (15) must hold with an equality sign throughout. The solution in (δ, V_1) is $(\delta^*(a_{P,1}),V_1(\mathbf{a}_P, \delta^*(a_{P,1})))$, with

$$\delta^*(a_{P,1}) = \frac{\pi_i^d(a_m) - \pi_m}{\pi_i^d(a_m) - \pi_i^d\,(a_{P,1})},$$

where the monotonicity of $\pi_i^d\,(a_{P,1})$ in $a_{P,1}$ (Lemma A-2) implies that $\delta^*(a_{P,1})$ is monotone non-decreasing in $a_{P,1}$. Next, introduce the constraint $(MLLC)$, which is equivalent to $a_{P,1} \succeq \underline{a}_P$. Then substitute $\underline{a}_P$ for $a_{P,1}$ to find $\delta^*(\underline{a}_P) = \underline{\delta}'$. ■

Proof of Remark 3. Recall from proof of Proposition 3 that $(MIC\,0)$ is written as $X\,(\delta) \leq V_0\,(\mathbf{a}_m, \delta) - V_0\,(\mathbf{a}_P, \delta)$, and (MPC) as $V_0\,(\mathbf{a}_m, \delta) - V_0\,(\mathbf{a}_P, \delta) \leq \overline{Y}\,(\delta)$, with $X\,(\delta) \equiv \left[\pi_i^d\,(a_m) - \pi_m\right]/\delta$ and $\overline{Y}\,(\delta) \equiv \pi_m/\,(1-\delta)$. If $(\mathbf{a}_P, \delta) = (\overline{\mathbf{a}}_P, \overline{\delta})$ and $\overline{a}_P \succ a_P^*$ we know that $X\left(\overline{\delta}\right) = V_0\left(\mathbf{a}_m, \overline{\delta}\right) - V_0\left(\overline{\mathbf{a}}_P, \overline{\delta}\right) = \overline{Y}\,(\delta)$, where $\overline{\mathbf{a}}_P \equiv (\overline{a}_P, a_m, \ldots, a_m)$, while all other multi-period incentive constraints are satisfied also. Given $\overline{\delta}$, consider a change from $\overline{\mathbf{a}}_P$ to $\overline{\mathbf{a}}'_P$, with $a'_{P,1} \succ \overline{a}_P$ and $a'_{P,k} \preceq a_m$ for some $k > 1$, that verifies $V_0\,(\overline{\mathbf{a}}_P, \delta) - V_0\,(\overline{\mathbf{a}}'_P, \delta) = 0$. For all $l > 1$, the continuity of $\pi(a_{P,k})$ in $a_{P,k}$ implies that the number of solutions $\overline{\mathbf{a}}'_P$ to the latter equation is infinite. By the very nature of the change both constraints $(MIC\,0)$ and (MPC) remain exactly satisfied, while by continuity $(MIC\,1)$ remains satisfied as well for a sufficiently small adjustment (it was slack for $a_{P,1} = \overline{a}_P$). Moreover, the $l-1$ remaining multi-period incentive constraints in $(MIC\,2\text{-}MIC\,l)$ are relaxed as a result of an adjustment from a_m "down" to $a'_{P,k} \prec a_m$ in any of the $k > 1$ following periods of punishment, all other things remaining equal. It follows that a_m is implementable if $(\mathbf{a}_P, \delta) = (\overline{\mathbf{a}}'_P, \overline{\delta})$. ■

In this appendix we introduce two additional technical results we need in order to prove Proposition 4.

Lemma A-4. *For all $\underline{V}$ verifying $\underline{\pi} < (1-\delta)\,\underline{V} \leq \pi_m$, there exists a finite l and a punishment $\underline{\mathbf{a}}_P \equiv (\underline{a}_P, a_{P,2}, \ldots, a_{P,k}, \ldots, a_{P,l})$, with $a_{P,k} \succeq \underline{a}_P$ for all $k > 1$, such that $V_1\,(\underline{\mathbf{a}}_P, \delta) = \underline{V}$.*

Proof. There are three steps: (1) we show that, given any δ, for any $l \geq 2$ there exists a punishment $\underline{\mathbf{a}}_P^l$ of length l such that $V_1\,(\underline{\mathbf{a}}_P, \delta) = \underline{V}$ for any $\underline{V}$ in a closed interval I_l we define; (2) we establish that the upper-bound of I_{l+1} is the lower bound of I_l so that their finite union

$I_L = \cup_{l=1}^{L} I_l$ is itself a closed interval; (3) we conclude by evidencing that the lower and upper bounds of the union of intervals are respectively $\underline{\pi}/(1-\delta)$ and $\pi_m/(1-\delta)$.

(1) Define $\mathbf{a}_P^l \equiv (a_{P,1}^l, a_{P,2}^l, \ldots, a_{P,k}^l, \ldots, a_{P,l}^l)$, where $a_{P,k}^l = \underline{a}_P$ for all $k = 1, 2, \ldots, l-1$, and $a_P^l \succeq \underline{a}_P$. Here firms opt for the most severe action $\underline{a}_P$ in the first $l-1$ periods, and for a possibly softer action in the l-th period. In the latter final period, the continuity of π in a_P^l implies that $\pi(a_P^l)$ may take any value in $[\pi(\underline{a}_P), \pi_m]$. Let $\underline{\mathbf{a}}_P^l$ and $\overline{\mathbf{a}}_P^l$ denote the just defined penal code $\mathbf{a}_P^l$ where $a_{P,l}^l = \underline{a}_P$ and $a_{P,l}^l = a_m$ respectively. By definition, for any value $\underline{V}$ in $I_l = [V_1(\underline{\mathbf{a}}_P^l, \delta), V_1(\overline{\mathbf{a}}_P^l, \delta)]$, there exists $\mathbf{a}_P^l$ such that $V_1(\mathbf{a}_P^l, \delta) = \underline{V}$.

(2) Clearly, $V_1(\underline{\mathbf{a}}_P^l, \delta) = V_1\left(\overline{\mathbf{a}}_P^{l+1}, \delta\right)$ so that $I_L = \cup_{l=1}^{L} I_l = [V_1(\underline{\mathbf{a}}_P^L, \delta), V_1(\overline{\mathbf{a}}_P^1, \delta)]$ for any integer $L > 1$.

(3) From the definition of continuation profits in (5) we know that $V_1(\overline{\mathbf{a}}_P^1, \delta) = \pi_m/(1-\delta)$, while $V_1(\underline{\mathbf{a}}_P^L, \delta)$ verifies

$$(1-\delta) V_1(\underline{\mathbf{a}}_P^L, \delta) = \underline{\pi} + \delta^{l-1}(\pi_m - \underline{\pi}).$$

Since $\lim_{L\to\infty}(\underline{\pi} + \delta^{L-1}(\pi_m - \underline{\pi})) = \underline{\pi}$, for any $\underline{V} > \underline{\pi}/(1-\delta)$ there exists a finite L such that $\underline{\pi} + \delta^{L-1}(\pi_m - \underline{\pi}) \leq (1-\delta)\underline{V}$ so that $\underline{V} \in [V_1(\underline{\mathbf{a}}_P^L, \delta), V_1(\overline{\mathbf{a}}_P^1, \delta)]$, and there exists a punishment profile $\mathbf{a}_P^l$, with $l \leq L$, such that $V_1(\mathbf{a}_P^l, \delta) = \underline{V}$. ■

We may now use Lemma A-4.

Proof of Proposition 4. As we are interested in establishing implementability for $\delta \geq \underline{\delta}_M \equiv \sup\{\overline{\delta}, \underline{\delta}'\}$, there are two cases that depend on the comparison of $\underline{\delta}'$ and $\overline{\delta}$. In both cases: (1) we establish that there exists a finite punishment, we denote $\mathbf{a}_P$, which is such that $V_1(\mathbf{a}_P, \delta)$ is equal to a particular value we explicit; (2) we check that all incentive constraints are satisfied; (3) we also verify that the participation and limited liability constraints hold.

($\underline{\delta}' \geq \overline{\delta} \Rightarrow \underline{\delta}_M = \underline{\delta}'$)

(1) Define implicitly $\underline{\mathbf{a}}_P$, specified to take the form of $\mathbf{a}_P^l$ as introduced in Lemma A-4 (so that $(MLLC)$ is satisfied) by

$$V_1(\underline{\mathbf{a}}_P, \delta) = \frac{1}{1-\delta}\left(\underline{\pi} + \frac{\pi_i^d(\underline{a}_P) - \underline{\pi}}{\delta}\right) \tag{20}$$

which describes continuation profits from the 2nd period of punishment onward.[31] Given δ, from Lemma A-4 a sufficient condition for $\underline{\mathbf{a}}_P$ to be well defined is $\underline{\pi} < (1-\delta) V_1(\underline{\mathbf{a}}_P, \delta) \leq \pi_m$. To

[31] In order to obtain the expression in (20), substitute $\underline{a}_P$ for $a_{P,1}$, and $\underline{\mathbf{a}}_P$ for $\mathbf{a}_P$, in $(MIC\,1)$ written with an equality sign, and reorganize terms.

check this holds, consider the two inequalities in turn: (i) We have $\underline{\pi} < (1-\delta) V_1 (\underline{\mathbf{a}}_P, \delta)$ since $\left[\pi_i^d (\underline{a}_P) - \underline{\pi}\right]/\delta > 0$ (by definition), for all $\delta > 0$. (ii) Toward $V_1 (\underline{\mathbf{a}}_P, \delta) \leq \pi_m/(1-\delta)$ first note that $\underline{a}_P \succeq a_P^*$ implies that $\pi_i^d(a_m) - \pi_m \geq \pi_i^d(\underline{a}_P) - \pi(\underline{a}_P)$ from Assumption $(A7)$ and Lemma A-1. From the expression of $\underline{\delta}'$, as displayed in Lemma 4, it follows that

$$\underline{\delta}' \leq \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \pi(\underline{a}_P)}. \tag{21}$$

Then pick $\delta = \underline{\delta}'$. Now (21), and $X\left(\underline{\delta}'\right) \equiv \left[\pi_i^d(a_m) - \pi_m\right]/\underline{\delta}' = V_0\left(\mathbf{a}_m, \underline{\delta}'\right) - V_0\left(\underline{\mathbf{a}}_P, \underline{\delta}'\right)$, imply that $\pi_m - \pi(\underline{a}_P) \leq V_0\left(\mathbf{a}_m, \underline{\delta}'\right) - V_0\left(\underline{\mathbf{a}}_P, \underline{\delta}'\right)$. Moreover, substituting $\left(1-\underline{\delta}'\right) V_0\left(\mathbf{a}_m, \underline{\delta}'\right)$ for π_m in the latter expression leads to $V_0\left(\underline{\mathbf{a}}_P, \underline{\delta}'\right) \leq \underline{\delta}' V_0\left(\mathbf{a}_m, \underline{\delta}'\right) + \pi(\underline{a}_P)$. Then substituting $\pi(\underline{a}_P) + \underline{\delta}' V_1\left(\underline{\mathbf{a}}_P, \underline{\delta}'\right)$ for $V_0\left(\underline{\mathbf{a}}_P, \underline{\delta}'\right)$ results in $\left(1-\underline{\delta}'\right) V_1\left(\underline{\mathbf{a}}_P, \underline{\delta}'\right) \leq \pi_m$, as needed. As $(1-\delta) V_1 (\underline{\mathbf{a}}_P, \delta)$ is monotone decreasing in δ, it follows that $(1-\delta) V_1 (\underline{\mathbf{a}}_P, \delta) \leq \pi_m$ for all $\delta \geq \underline{\delta}'$. Eventually, (i) and (ii) establish that there exists at least one $\underline{\mathbf{a}}_P$ for a finite l such that $V_1 (\underline{\mathbf{a}}_P, \delta)$ satisfies (20) for all $\delta \geq \underline{\delta}'$.

(2) Recall from the proof of Lemma 4 that for collusion to be implemented at $\delta = \underline{\delta}'$, it must be the case that the two constraints $(MIC\,0)$ and $(MIC\,1)$ are binding and that $a_{P,1} = \underline{a}_P$. Therefore, for $\mathbf{a}_P = \underline{\mathbf{a}}_P \equiv (\underline{a}_P, \underline{a}_P, \ldots, \underline{a}_P, a_{P,l})$, where $a_{P,l} \succeq \underline{a}_P$ (i.e., the same vector as introduced in the proof of Lemma A-4), if $\delta = \underline{\delta}'$ we have that $(MIC\,0$-$MIC\,1)$ are exactly satisfied. Clearly, $V_{k+1} (\underline{\mathbf{a}}_P, \delta)$ is strictly increasing in k as long as $1 \leq k \leq l-1$. Since $\pi_i^d\left(\underline{a}_{P,k}\right) - \pi(\underline{a}_{P,k})$ is identical for all $1 \leq k \leq l-1$, if $(MIC\,1)$ holds and is binding, it must be the case that all constraints $(MIC\,2), \ldots, (MIC\,l-1)$ hold also and are slack. Finally, to check that the last incentive constraint $(MIC\,l)$ is also satisfied, we compare it with $(MIC\,0)$. First, observe that the terms on the RHS of the inequality sign are the same in the two constraints, because $V_l (\mathbf{a}_P, \delta) = V_0 (\mathbf{a}_m, \delta)$, all $\mathbf{a}_P$. Next, consider the terms on the LHS of the inequality side of $(MIC\,l)$. There is no loss of generality in assuming that $a_{P,l} \prec a_m$. (If there is equality, collusion can be implemented by the means of a $l-1$ punishment scheme where $a_{P,l-1} = \underline{a}_P \prec a_m$). Assuming this is the case, we know from Assumption $(A7)$ and Lemma A-1 that $\pi_i^d(a_{P,l}) - \pi(a_{P,l}) < (=) \pi_i^d(a_m) - \pi_m$, for all $a_{P,l} \succ (=) a_P^*$ (as in the present case, since here $a_{P,l} \succeq \underline{a}_P \succeq a_P^*$). Therefore, if $(MIC\,0)$ holds and is binding, it must be the case that $(MIC\,l)$ holds also and is slack. This says that, in the absence of participation constraint, a_m is implementable with at least one l-punishment vector, that is $\mathbf{a}_P = \underline{\mathbf{a}}_P$, when $\delta = \underline{\delta}'$. Since for $\mathbf{a}_P = \underline{\mathbf{a}}_P$ all MICs $(MIC\,0$-$MIC\,l)$ are monotone increasing in δ, this also holds for all $\delta \geq \underline{\delta}'$.

(3) Consider now the participation constraint. If $\overline{\delta} \leq \underline{\delta}'$, then the comparison of the developed expressions for the two thresholds implies that $\pi_i^d(a_m) - \pi_i^d(\underline{a}_P) \leq \pi_m - \overline{\pi}$. Since $\overline{\pi} = \pi_m - \pi_i^d(a_m)$

by definition, we have $\pi_i^d(\underline{a}_P) \geq 0$. Since $V_0(\underline{\mathbf{a}}_P, \delta) = \underline{\pi} + \delta V_1(\underline{\mathbf{a}}_P, \delta)$, with $V_1(\underline{\mathbf{a}}_P, \delta)$ as in (20), and $\underline{\mathbf{a}}_P$ as defined above in (1), we have $V_0(\underline{\mathbf{a}}_P, \delta) \geq 0$, which says that the participation constraint (MPC) is also satisfied for $\mathbf{a}_P = \underline{\mathbf{a}}_P$ and $\delta \geq \underline{\delta}'$. This says that a_m is implementable with a finite punishment scheme for all $\delta \geq \underline{\delta}'$. Then recall from Lemma 4 that the lowest δ compatible with $(MIC\,0$-$MIC\,1)$ and $(MLLC)$ is $\underline{\delta}'$. It follows that $\underline{\delta}'$ is the lowest possible discount factor that implements a_m.

$(\overline{\delta} > \underline{\delta}' \Rightarrow \underline{\delta}_M = \overline{\delta})$

(1) We proceed as in the previous case to define implicitly $\bar{\mathbf{a}}_P$ by

$$V_1(\bar{\mathbf{a}}_P, \delta) = -\frac{\underline{\pi}}{\delta}. \tag{22}$$

Again, we must check that $\bar{\mathbf{a}}_P$ satisfies the sufficient condition introduced in Lemma A-4, that is $\underline{\pi} < -\frac{(1-\delta)}{\delta}\underline{\pi} \leq \pi_m$, for all $\delta \geq \overline{\delta}$.[32] The LHS inequality is always satisfied for $\delta \in (0,1]$. On the RHS, $\underline{a}_P \succeq \overline{a}_P$ implies that $\underline{\pi} \geq \overline{\pi} = \pi_m - \pi_i^d(a_m)$, recalling that the latter equality is the implicit definition of $\overline{a}_P$. As a result $-\frac{(1-\overline{\delta})}{\overline{\delta}}\underline{\pi} \leq \pi_m$, which extends to any $\delta \geq \overline{\delta}$ by monotonicity. Hence there exists at least one $\bar{\mathbf{a}}_P$ for a finite l such that $V_1(\bar{\mathbf{a}}_P, \delta)$ satisfies (22) for any $\delta \geq \overline{\delta}$.

(2) At $\delta = \overline{\delta}$, we check that $(MIC\,0$-$MIC\,1)$ are satisfied for $\mathbf{a}_P = \bar{\mathbf{a}}_P \equiv (\overline{a}_P, a_m, \ldots, a_m)$, so that $\overline{a}_{P,1} = \underline{a}_P$, and $(MLLC)$ is satisfied by construction). Indeed $X(\overline{\delta}) = V_0(\mathbf{a}_m, \overline{\delta}) - V_0(\bar{\mathbf{a}}_P, \overline{\delta}) < Y(\overline{\delta}, \underline{a}_P)$ with $X(\overline{\delta}) = \pi_i^d(a_m)$, $Y(\overline{\delta}, \underline{a}_P) = \pi_i^d(a_m)\left(1 - \frac{\pi_i^d(\underline{a}_P)}{\pi_m}\right) > \pi_i^d(a_m)$ since $\pi_i^d(\underline{a}_P) < 0$, and $V_0(\mathbf{a}_m, \overline{\delta}) - V_0(\bar{\mathbf{a}}_P, \overline{\delta}) = V_0(\mathbf{a}_m, \overline{\delta}) = \frac{\pi_m}{1-\overline{\delta}} = \pi_i^d(a_m)$. Again, $V_{k+1}(\bar{\mathbf{a}}_P, \delta)$ is strictly increasing in k as long as $1 \leq k \leq l-1$. Since $\pi_i^d(a_{P,k}) - \pi(a_{P,k})$ is identical for all $1 \leq k \leq l-1$, if $(MIC\,1)$ is satisfied, it must be the case all constraints $(MIC\,1), \ldots, (MIC\,l-1)$ are also satisfied. As for the last incentive constraint, that is $(MIC\,l)$, we compare it with $(MIC\,0)$. The terms on the RHS of the inequality sign are the same in the two constraints, because $V_l(\mathbf{a}_P, \delta) = V_0(\mathbf{a}_m, \delta)$, all $\mathbf{a}_P$. On the LHS, there is no loss of generality in assuming that $a_{P,l} \prec a_m$. (If there is equality, collusion can be implemented by the means of a $l-1$ punishment scheme where $a_{P,l-1} = \underline{a}_P \prec a_m$). Assuming this is the case, we know from Assumption $(A7)$ and Lemma A-1 that $\pi_i^d(a_{P,l}) - \pi(a_{P,l}) < \pi_i^d(a_m) - \pi_m$, for all $a_{P,l} \succ (=) a_P^*$, as in the present case. Therefore, if $(MIC\,0)$ holds and is binding, it must be the case that $(MIC\,l)$ holds also and is slack. We obtain that all incentive constraints are satisfied. Again, since for $\mathbf{a}_P = \bar{\mathbf{a}}_P$ all $MICs$ $(MIC\,0$-$MIC\,l)$ are monotone increasing in δ, this also holds for all $\delta \geq \overline{\delta}$.

[32] In order to obtain the expression in (22), substitute $\underline{a}_P$ for $a_{P,1}$, and $\bar{\mathbf{a}}_P$ for $\mathbf{a}_P$, in (MPC) written with an equality sign for $s = 0$, and reorganize terms.

(3) By construction, from (22), $V_0(\bar{\mathbf{a}}_P, \delta) = 0$ hence (MPC) is satisfied for all δ. Given the structure of $\bar{\mathbf{a}}_P$, $(MLLC)$ is also satisfied. This says that a_m is implementable with a finite punishment scheme for all $\delta \geq \overline{\delta}$. Then recall from Lemma 3 that the lowest δ compatible with $(MIC\,0$-$MIC\,1)$ and (MPC) is $\overline{\delta}$. It follows that $\overline{\delta}$ is the lowest possible discount factor that implements a_m. ■

Proof of Remark 4. Consider again the punishment profile of Lemma A-4, that is $\mathbf{a}_P^l \equiv (a_{P,1}^l, a_{P,2}^l, \ldots, a_{P,k}^l, \ldots, a_{P,l}^l)$, where $a_{P,k}^l = \underline{a}_P$ for all $k = 1, 2, \ldots, l-1$, and $a_{P,l}^l \succeq \underline{a}_P$. We know from Proposition 4 that there exists a punishment profile of this kind that allows firms to implement a_m for $\delta = \underline{\delta}_M$. We also have shown that, for this punishment profile, the $(MIC\,l)$ constraint holds and is slack. One may construct a $l+1$ period punishment profile identical to $\mathbf{a}_P^l$ up to the period $k = l-1$ and with $\mathring{a}_{P,l} \succ a_{P,l}^l$ and $\mathring{a}_{P,l+1} \prec a_m$ such that

$$\pi(a_{P,l}) + \delta\pi_m = \pi(\mathring{a}_{P,l}) + \delta\pi(\mathring{a}_{P,l+1})$$

and all incentive constraints are satisfied. ■

Proof of Proposition 5. It is assumed that $\underline{a}_P \succ a_P^*, \overline{a}_P$. To see that $\underline{\delta}_M < \underline{\delta}$, recall that $\underline{\delta}_M \equiv \sup\{\underline{\delta}', \overline{\delta}\}$ and consider the two possible cases: (i) If $\underline{\delta}_M = \overline{\delta}$ then it suffices to recall that $\underline{a}_P \succ a_P^*, \overline{a}_P$ implies $\underline{\delta} > \overline{\delta}$ (see Remark 1) to conclude. (ii) If $\underline{\delta}_M = \underline{\delta}'$ then compare the expressions of the denominators of $\underline{\delta}'$ and $\underline{\delta}$. We have $\pi_i^d(a_m) - \pi_i^d(\underline{a}_P) > \pi_m - \underline{\pi}$ if and only if $\pi_i^d(a_m) - \pi_m > \pi_i^d(\underline{a}_P) - \underline{\pi}$. To establish the latter property, recall from Assumption $(A7)$ that there exists a unique $\breve{a} \prec a_m$ such that $\pi_i^d(a_m) - \pi_m = \pi_i^d(\breve{a}) - \pi(\breve{a})$, and from Lemma A-1 that $\breve{a} = a_P^*$. Therefore, here $\underline{a}_P \neq a_P^*$ implies that either $\pi_i^d(a_m) - \pi_m < \pi_i^d(\underline{a}_P) - \underline{\pi}$, which is not possible (the incentive to deviate from a_m is an upper bound to a firm's incentive to deviate from any a, see comment below Lemma A-1), or $\pi_i^d(a_m) - \pi_m > \pi_i^d(\underline{a}_P) - \underline{\pi}$, which thus holds.

For the comparison of $\underline{\delta}_M$ with the other discount thresholds, there are two cases. Suppose first that $a_P^* \succeq \overline{a}_P$, to compare $\underline{\delta}_M$ with δ^* (regime 1). From Remark 2 we obtain directly that $\delta^* < \underline{\delta}_M$. Suppose next that $\overline{a}_P \succ a_P^*$, to compare $\underline{\delta}_M$ with $\overline{\delta}$ (regime 2). From the definition of $\underline{\delta}_M$ we obtain directly that $\underline{\delta}_M \geq \overline{\delta}$. Finally, to demonstrate that $\underline{\delta}_M = \overline{\delta}$ if and only if $\tilde{a}_P \succeq \underline{a}_P \succ \overline{a}_P \succ a_P^*$, note that $\tilde{a}_P \succeq \underline{a}_P$ if and only if $\pi_i^d(\underline{a}_P) \leq 0$ from Assumption $(A5)$, and equivalently $\pi_i^d(a_m) - \pi_i^d(\underline{a}_P) \geq \pi_i^d(a_m)$. Recalling that $\pi_m - \overline{\pi} = \pi_i^d(a_m)$ by (implicit) definition of $\overline{a}_P$, it follows that $\tilde{a}_P \succeq \underline{a}_P$ if and only if $\overline{\delta} \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_m - \overline{\pi}} = \frac{\pi_i^d(a_m) - \pi_m}{\pi_i^d(a_m)} \geq \underline{\delta}' \equiv \frac{\pi_i^d(a_m) - \pi_m}{\pi_i^d(a_m) - \pi_i^d(\underline{a}_P)}$, establishing that $\underline{\delta}_M = \overline{\delta}$ (by definition), as needed. ■

8.3 A Linear Example

In this appendix we compute the specific algebraic expressions we need for the analysis of the linear example in Section 5. Inverse demand functions for firm i and all other symmetric firms j are given by (9) and (10). Therefore symmetric profits are

$$\pi(q) = \begin{cases} (1 - q(1 + \gamma(n-1)) - c)q \text{ if } q \leq \underline{q}_P \equiv \frac{1}{1+\gamma(n-1)} \\ -cq \text{ if } q \geq \underline{q}_P \end{cases}, \tag{23}$$

where the piecewise structure results from the non-negativity constraint we impose on prices (solve $1 - q_i - \gamma(n-1)q_j \geq 0$ for $q_i = q_j = q$ to find $q \leq \underline{q}_P \equiv \frac{1}{(1+\gamma(n-1))}$). The collusive quantity and corresponding profits are $q_m^* = \frac{1-c}{2(1+\gamma(n-1))}$ and $\pi_m^* = \frac{(1-c)^2}{4(1+\gamma(n-1))}$, respectively (there is perfect collusion, with $\pi_m^* \equiv \pi(q_m^*)$). The one-shot best deviation profits are

$$\pi_i^d(q) = \begin{cases} \frac{1}{4}(1 - c - \gamma(n-1)q)^2 \text{ if } q \leq \tilde{q}_P \equiv \frac{1-c}{\gamma(n-1)} \\ 0 \text{ otherwise} \end{cases}, \tag{24}$$

where $\tilde{q}_P$ is the solution to $\pi_i^d(q) = 0$ (here $f = 0$ implies $\tilde{q}_P = \hat{q}_P$, see $(A5)$ and $(A6)$). Since $q_m^* < \tilde{q}_P$ for all parameter values, firm i's best-reply profits, when each firm in $N \backslash \{i\}$ sells q_m^*, are $\pi_i^d(q_m^*) = \frac{(1-c)^2}{16} \frac{(\gamma(n-1)+2)^2}{(1+\gamma(n-1))^2}$, from (24).

8.3.1 Implementability (feasibility of collusion)

In the single-period benchmark set-up (Section 3), consider the expression of $\underline{\delta}$ in (2). We have $\underline{\delta} \leq 1$ if and only if $c \geq \tilde{c}$, where

$$\tilde{c} \equiv \frac{\gamma^2(n-1)^2 + 4(\gamma(n-1)+1) - 4\sqrt{\gamma^2(n-1)^2(1+\gamma(n-1))}}{\gamma^2(n-1)^2 - 4(\gamma(n-1)+1)},$$

which is the only admissible root to $\underline{\pi} = \pi_m^* - \left(\pi_i^d(q_m^*) - \pi_m^*\right)$, the second root being negative for all γ, n. Given $n \geq 2$, we have $\tilde{c} \geq 0$ if on and only if $\gamma \geq \check{\gamma} \equiv 2\frac{1+\sqrt{2}}{n-1}$. Note that $\inf\{\check{\gamma}, 1\} = 1$ if and only if $n < 6$. Hence $\tilde{c}$ is positive only when $n \geq 6$. In that case, q_m^* cannot be implemented with a single-period scheme for all $c < \tilde{c}$ (see Fig. 1).

Now we check that q_m^* can always be implemented for some δ sufficiently high in the multi-period punishment case (Section 4). Recall that $\underline{\delta}_M \equiv \sup\{\underline{\delta}', \overline{\delta}\}$. We know from Lemma 3 that $\overline{\delta} < 1$ for all $\pi_m^* > 0$, and from Lemma 4 that $\underline{\delta}' < 1$ if and only if $\pi_i^d\left(\underline{q}_P\right) < \pi_m^*$. Then from (24)

there are two cases: if $c > \frac{1}{1+\gamma(n-1)}$, or equivalently $\underline{q}_P > \tilde{q}_P$, we have $\pi_i^d \left(\underline{q}_P\right) = 0 < \pi_m^*$ for all $\pi_m^* > 0$; otherwise $\pi_i^d \left(\underline{q}_P\right) < \pi_m^*$ if and only if $0 \leq c \leq \tilde{c}'$, where

$$\tilde{c}' \equiv \frac{1}{\sqrt{1+\gamma(n-1)}}.$$

This is the only admissible root to $\pi_i^d \left(\underline{q}_P\right) = \pi_m^*$, the second root being negative for all γ, n. Then it is sufficient to observe that $\tilde{c}' > \frac{1}{1+\gamma(n-1)}$ to verify that $\pi_i^d \left(\underline{q}_P\right) < \pi_m^*$.

8.3.2 Calculation of the discount thresholds

For all $q_P > q_m^*$ one must consider the two forms of $\pi_i^d(q_P)$, that depend on the comparison of q_P with $\tilde{q}_P$. This leads to two cases:

(1) If $q_P \leq \tilde{q}_P \equiv \frac{1-c}{\gamma(n-1)}$ best-reply profits are $\pi_i^d (q_P) = \frac{1}{4} \left(1 - c - \gamma(n-1)\right) q)^2$ and (PC) is slack. When only $(IC0)$ and $(IC1)$ are considered, we know (from Lemma 1) that the optimal punishment q_P^* is a solution in q_P of $\pi_i^d (q_P) - \pi (q_P) = \pi_i^d (q_m^*) - \pi_m^*$. The only two solutions are q_m^*, which does not apply as a punishment; the other one is

$$q_P^* = \frac{1-c}{2} \frac{3\gamma(n-1)+2}{[2+\gamma(n-1)]\,[1+\gamma(n-1)]}.$$

The latter punishment quantity is defined only when lower than $\tilde{q}_P$, which holds if and only if $\gamma \leq \inf\{\check{\gamma}, 1\}$, recalling that $\check{\gamma} \equiv 2\frac{1+\sqrt{2}}{n-1}$ and $\inf\{\check{\gamma}, 1\} = 1$ if and only if $n < 6$. The threshold value for δ is

$$\delta^* = \frac{1}{16} \frac{[2+\gamma(n-1)]^2}{1+\gamma(n-1)} < 1. \tag{25}$$

This is Regime 1 (see (2)). Next, we find $q_P^* \leq \underline{q}_P$, so that the price $p_i(q_P^*, q_P^*)$ is non-negative and (LLC) is slack if and only if $c \geq \underline{c}'$, with

$$\underline{c}' \equiv \frac{\gamma(n-1)-2}{3\gamma(n-1)+2}. \tag{26}$$

The frontier $\underline{c}'$ intersects the line $c = 0$ from below at $\gamma = \hat{\gamma} \equiv \frac{2}{n-1}$. Therefore there exists $\underline{c}' > 0$ if and only if $\frac{2}{n-1} < 1$ (one checks that $\hat{\gamma} < \check{\gamma}$ for all $n \geq 2$), or equivalently $n > 3$, otherwise $\underline{c}' = 0$ for all parameter values. Whenever $c < \underline{c}'$ we have $\underline{q}_P < q_P^* \leq \tilde{q}_P$ and (LLC) binds. (Here $q_P^* \leq \tilde{q}_P$ is implied by $\gamma \leq \inf\{\check{\gamma}, 1\}$.) This is regime 3.

(2) If $q_P > \tilde{q}_P \equiv \frac{1-c}{\gamma(n-1)}$ best-reply profits are $\pi_i^d(q_P) = 0$ and $(IC1)$ is identically equal to (PC). (This holds because $f = 0$, otherwise $f > 0$ would imply that $(IC1)$ is strictly weaker than (PC).) It follows from the previous case (where $q_P \leq \tilde{q}_P \equiv \frac{1-c}{\gamma(n-1)}$) that we need only consider $\gamma \geq \check{\gamma}$ and $n \geq 6$ to complete the analysis. There are two solutions in q_P to $-\pi(q_P) = \pi_i^d(q_m^*) - \pi_m^*$, the equation that defines $\overline{q}_P$ implicitly. The first one is strictly less than $\tilde{q}_P$ for all $c < 1$, therefore it is not admissible; the second one is then

$$\overline{q}_P = \frac{1-c}{4}\frac{2[1+\gamma(n-1)] + [2+\gamma(n-1)]\sqrt{1+\gamma(n-1)}}{[1+\gamma(n-1)]^2},$$

which we check is always strictly higher than $\tilde{q}_P$. Then the threshold value for δ now is

$$\overline{\delta} = \left(\frac{\gamma(n-1)}{2+\gamma(n-1)}\right)^2 < 1. \tag{27}$$

This is Regime 2 (see (2)). Next, we find $\overline{q}_P < (=)\underline{q}_P$, so that the price $p_i(\overline{q}_P, \overline{q}_P)$ is non-negative and (LLC) is slack if and only if $c > (=)\underline{c}''$, with

$$\underline{c}'' \equiv \frac{\sqrt{1+\gamma(n-1)}\,[2+\gamma(n-1)] - 2[1+\gamma(n-1)]}{\sqrt{1+\gamma(n-1)}\,[2+\gamma(n-1)] + 2[1+\gamma(n-1)]}. \tag{28}$$

The frontier $\underline{c}''$ intersects from below the line $c = 0$ if $\gamma = 0$, and $\underline{c}'' > 0$ otherwise. Therefore $\underline{c}'' > 0$ for all $\gamma \geq \check{\gamma}$. Whenever $c < \underline{c}''$ we have $\underline{q}_P < \overline{q}_P \leq q_P^*$ and (LLC) binds. (Here $\overline{q}_P \leq q_P^*$ is implied by $\gamma \geq \check{\gamma}$ and $n \geq 6$.) This is regime 3.

The two preceding paragraphs delineate the parameter subsets in which regimes 1 and 2 apply, respectively. (In the latter case, since $f = 0$, note that $(IC1)$ being identical to (PC) implies that regimes 1 and 2 coincide for all points (n, γ, c) verifying $n \geq 6$, $\check{\gamma} \leq \gamma \leq 1$, and $\underline{c}'' \leq c < 1$.) All points in the parameter set where regime 3 applies were also identified. In the latter regime, the discount threshold $\underline{\delta}$ solves $\pi_i^d(q_m^*) - \pi_m^* = \delta\left(\pi_m^* - \pi\left(\underline{q}_P\right)\right)$. As the specific algebraic form of the latter expression does not depend on parameter values, for all n, γ, c there is a unique

$$\underline{\delta} = \frac{1}{4}\left(\frac{1-c}{1+c}\right)^2 \frac{(n-1)^2\gamma^2}{1+\gamma(n-1)}. \tag{29}$$

It remains to compute $\underline{\delta}_M$, the discount threshold when (LLC) binds and firms design the optimal l-period punishment scheme. We know (from Proposition 4) that $\underline{\delta}_M = \sup\{\underline{\delta}', \overline{\delta}\}$. Again we know from (24) there are two cases: 1) if $\underline{q}_P < \tilde{q}_P$, or equivalently $c < \frac{1}{1+\gamma(n-1)}$, we have $\pi_i^d\left(\underline{q}_P\right) = \frac{1}{4}\left(1-c-\gamma(n-1)\,\underline{q}_P\right)^2$, which implies that

$$\underline{\delta}_M = \frac{\gamma(n-1)(1-c)^2}{(1+c)[4(1-c) - \gamma(n-1)(3c-1)]} > \overline{\delta}; \tag{30}$$

and 2) if $\underline{q}_P \geq \tilde{q}_P$, or equivalently $c \geq \frac{1}{1+\gamma(n-1)}$, we have $\pi_i^d\left(\underline{q}_P\right) = 0$, hence

$$\underline{\delta}_M = \left(\frac{\gamma(n-1)}{2+\gamma(n-1)}\right)^2, \tag{31}$$

which is the same expression as $\overline{\delta}$ (regime 2), an illustration of Proposition 5.

8.3.3 Partition of the parameter space

Finally, we may now use the obtained algebraic expressions to describe extensively the partition of the parameter space (c, n, γ) in three subsets where either Regime 1, 2, or 3 apply, as follows:

1) Regime 1 applies if and only if
(i) $2 \leq n \leq 3;\ 0 \leq \gamma \leq 1;\ 0 \leq c < 1$; or
(ii) $4 \leq n \leq 5;\ 0 \leq \gamma \leq 1;\ \underline{c}' \leq c < 1$; or
(iii) $6 \leq n;\ 0 \leq \gamma \leq \hat{\gamma};\ 0 \leq c < 1$; or
(iv) $6 \leq n;\ \hat{\gamma} \leq \gamma \leq \check{\gamma};\ \underline{c}' \leq c < 1$.

2) Regime 2 applies if and only if
$6 \leq n;\ \check{\gamma} \leq \gamma \leq 1;\ \underline{c}'' \leq c < 1$.

3) Regime 3 applies if and only if
(i) $n = 3;\ \gamma = \hat{\gamma} = 1;\ c = \underline{c} = 0$; or
(ii) $4 \leq n \leq 5;\ \hat{\gamma} \leq \gamma \leq 1;\ 0 \leq c \leq \underline{c}'$; or
(iii) $6 \leq n;\ \hat{\gamma} \leq \gamma \leq \check{\gamma};\ 0 \leq c \leq \underline{c}'$; or
(iv) $6 \leq n;\ \check{\gamma} \leq \gamma \leq 1;\ 0 \leq c \leq \underline{c}''$.

At any point (n, γ, c) where Regime 3 applies (the grey area in Fig. 1), q_m^* is implementable for all $\delta \geq \underline{\delta}_M$. This illustrates Proposition 4. Next, as an illustration of Proposition 5, the grey area can be partitioned into three subsets, which describe the consequences of introducing a multi-period punishment scheme. For all points below the frontier $\underline{c}''$ and above the frontier $\tilde{q}_P = \underline{q}_P$ (so that $\tilde{q}_P \leq \underline{q}_P$ together with $f = 0$ imply $\pi_i^d\left(\underline{q}_P\right) = 0$), we have $\underline{\delta}_M = \overline{\delta}$. Then firms may implement a_m for all $\delta \geq \underline{\delta}_M = \overline{\delta}$ with a multi-period punishment. Second, in the grey area below the frontier $\tilde{q}_P = \underline{q}_P$ (in which case $\tilde{q}_P > \underline{q}_P$ implies $\pi_i^d\left(\underline{q}_P\right) > 0$) and for $\gamma \geq \check{\gamma}$, we have $\underline{\delta}_M > \overline{\delta}$. In that case firms cannot implement a_m for a discount level as low as $\overline{\delta}$. Eventually, for $\gamma < \check{\gamma}$ and below $\underline{c}'$, we have $\underline{q}_P < q_P^* \leq \overline{q}_P$, hence $\delta^* < \underline{\delta}_M < \underline{\delta}$. In words, the limited liability constraint binds, and several punishments with discounting are only an imperfect substitute for more severity in the first period. The same figure also helps identifying the role of fixed costs.

When $f = 0$, one can check that $(IC1)$ simplifies to the same expression as (PC). This does not hold whenever $f > 0$.[33] In that case all incentive constraints, together with the limited liability constraint, remain unchanged. The only difference is that the future stream of profits earned from the first period of punishment onward is reduced by the magnitude of fixed costs, so that the participation constraint becomes stronger. Hence the parameter subset where regime 2 applies expands. This has no impact on δ^*, $\overline{\delta}$, and $\underline{\delta}$, which correspond to each regime. ■

8.4 Policy Implications

Proof of Lemma 5. First, we observe that a_{NE} is implementable for $\underline{a}_R = a_{NE}$. Second, we show that, if an implementable $a_m \succ a_{NE}$ exists, it cannot be equal to $\underline{a}_R \succ a_{NE}$.

(1) Set $a_m = \underline{a}_R = a_{NE}$. For any $\delta \in (0, 1)$, given that a_{NE} is a stage-game equilibrium, it can be implemented repeatedly in all periods. (The participation constraint (PC) is also satisfied for all $\pi(a_{NE}) \geq 0$, the regulatory constraint (R) is exactly satisfied, as the two incentive constraints $(IC0\text{-}IC1)$ of the single period punishment scheme with $a_P = a_{NE}$.)

(2) Given δ, suppose that there exists an implementable $a_m = \underline{a}_R$, for a given $\underline{a}_R \succ a_{NE}$. We look for a contradiction. In that case, recall that $V_0(\mathbf{a}_P, \delta)$ was defined in (5) as the continuation profit earned by a firm from the first period of punishment onward. Then, the regulatory constraint (R) implies that any punishment profile $\mathbf{a}_P$ must satisfy $V_0(\mathbf{a}_P, \delta) \geq \frac{\pi(\underline{a}_R)}{1-\delta}$. Similarly, the continuation profit with collusion is $V_0(\mathbf{a}_m, \delta) = \frac{\pi_m}{1-\delta}$. It follows that $\mathbf{a}_P$ must satisfy $V_0(\mathbf{a}_m, \delta) - V_0(\mathbf{a}_P, \delta) \leq \frac{1}{1-\delta}(\pi_m - \pi(\underline{a}_R))$. Moreover, recall from the first multi-period incentive constraint (3) that $\mathbf{a}_P$ must also satisfy $\pi_i^d(a_m) - \pi_m \leq \delta [V_0(\mathbf{a}_m, \delta) - V_0(\mathbf{a}_P, \delta)]$. It follows that, for $a_m \succ a_{NE}$ to be implementable with $\mathbf{a}_P \succeq (\underline{a}_R, \ldots, \underline{a}_R)$ as a punishment profile, it must be the case that

$$\pi_i^d(a_m) - \pi_m \leq \frac{\delta}{1-\delta}(\pi_m - \pi(\underline{a}_R)). \tag{32}$$

This condition is violated when a_m approaches $\underline{a}_R$, for all $\underline{a}_R \succ a_{NE}$. Indeed, on the LHS $\lim_{a_m \to \underline{a}_R}\left(\pi_i^d(a_m) - \pi_m\right) > 0$, while on the RHS $\lim_{a_m \to \underline{a}_R}(\pi_m - \pi(\underline{a}_R)) = 0$. Hence necessarily $a_m \succ \underline{a}_R$. ■

[33] If $f = 0$ we have $\pi_i^d(q_P) = -f = 0$ for all $q_P \geq \tilde{q}_P = \hat{q}_P$. In that case, the solution to the δ-minimization problem in q_P, under $(IC0)$ and $(IC1)$ only, is the same as the solution under $(IC0)$ and (PC). If $f > 0$ the constraint (PC) becomes stronger than $(IC1)$ for all $q_P \geq \tilde{q}_P$, with $\hat{q}_P > \tilde{q}_P$ (see assumptions $(A5)$ and $(A6)$). We may also assume that $f < 0$ to capture the existence of a profitable outside option. In this case (PC) is weaker than with a non-negative fixed cost.

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